

HOMOTOPY INVARIANT PRESHEAVES WITH FRAMED TRANSFERS

GRIGORY GARKUSHA AND IVAN PANIN

ABSTRACT. The category of framed correspondences $Fr_*(k)$, framed presheaves and framed sheaves were invented by Voevodsky in his unpublished notes [6]. Based on the theory, framed motives are introduced and studied in [3]. The main aim of this paper is to prove the following result, which was first announced in [3].

For any \mathbb{A}^1 -invariant quasi-stable additive framed presheaf of abelian groups \mathcal{F} , the associated Nisnevich sheaf \mathcal{F}_{nis} is \mathbb{A}^1 -invariant provided that the base field k is infinite. Moreover, if the base field k is perfect and infinite then every \mathbb{A}^1 -invariant quasi-stable Nisnevich framed sheaf is strictly \mathbb{A}^1 -invariant.

This result and the paper are inspired by Voevodsky's paper [7].

CONTENTS

1. Introduction	1
2. A few theorems	2
3. Notation and agreements	7
4. Some homotopies	8
5. Injectivity and excision on affine line	10
6. Excision on relative affine line	12
7. Injectivity for local schemes	12
8. Preliminaries for the injective part of the étale excision	15
9. Reducing Theorem 2.13 to Propositions 8.6 and 8.9	17
10. Preliminaries for the surjective part of the étale excision	20
11. Reducing Theorem 2.14 to Propositions 10.1 and 10.5	22
12. Two useful theorems	26
13. Construction of h'_θ , F and h_θ from Propositions 10.5 and 8.9	28
14. Homotopy invariance of cohomology presheaves	31
References	36

1. INTRODUCTION

The main result can be restated in terms of $\mathbb{Z}F_*$ -presheaves of abelian groups on smooth varieties. Recall that $\mathbb{Z}F_*(k)$ is defined in [3] as an additive category whose objects are those of

2010 *Mathematics Subject Classification.* 14F42, 14F05.

Key words and phrases. Motivic homotopy theory, framed presheaves.

Theorem 1.1 is proved thanks to the support of the Russian Science Foundation (grant no. 14-21-00035).

The second author thanks the Institute for Advanced Study for the support and for the kind hospitality during his visit in the Second Term of 2014–2015.

Sm/k and Hom-groups are defined as follows. We set for every $n \geq 0$ and $X, Y \in Sm/k$,

$$\mathbb{Z}F_n(X, Y) = \mathbb{Z}Fr_n(X, Y) / \langle Z_1 \sqcup Z_2 - Z_1 - Z_2 \rangle,$$

where Z_1, Z_2 are supports of correspondences. In other words, $\mathbb{Z}F_n(X, Y)$ is a free abelian group generated by the framed correspondences of level n with connected supports. We then set

$$\mathrm{Hom}_{\mathbb{Z}F_*(k)}(X, Y) = \bigoplus_{n \geq 0} \mathbb{Z}F_n(X, Y).$$

The canonical morphisms $Fr_*(X, Y) \rightarrow \mathrm{Hom}_{\mathbb{Z}F_*(k)}(X, Y)$ define a functor $R : Fr_*(k) \rightarrow \mathbb{Z}F_*(k)$, which is the identity on objects. For any \mathbb{A}^1 -invariant quasi-stable $\mathbb{Z}F_*$ -presheaf of abelian groups \mathcal{F} the functor $\mathcal{F} \circ R : Fr_*(k)^{\mathrm{op}} \rightarrow Ab$ is \mathbb{A}^1 -invariant quasi-stable radditive framed presheaf of abelian groups.

By definition, a Fr_* -presheaf \mathcal{F} of abelian groups is stable if for any k -smooth variety the pull-back map $\sigma_X^* : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$ equals the identity map, where $\sigma_X = (X \times 0, X \times \mathbb{A}^1, t; pr_X) \in Fr_1(X, X)$. In turn, \mathcal{F} is quasi-stable if for any k -smooth variety the pull-back map $\sigma_X^* : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$ is an isomorphism. Also, recall that \mathcal{F} is radditive if $\mathcal{F}(\emptyset) = \{0\}$ and $\mathcal{F}(X_1 \sqcup X_2) = \mathcal{F}(X_1) \times \mathcal{F}(X_2)$.

For any \mathbb{A}^1 -invariant stable radditive Fr_* -presheaf of abelian groups G there is a unique \mathbb{A}^1 -invariant stable $\mathbb{Z}F_*$ -presheaf of abelian groups \mathcal{F} such that $G = \mathcal{F} \circ R$. This follows easily from the additivity theorem [3, Theorem 5.1].

That is the category of \mathbb{A}^1 -invariant stable radditive framed presheaves of abelian groups is equivalent to the category of \mathbb{A}^1 -invariant stable $\mathbb{Z}F_*$ -presheaves of abelian groups.

The latter means that the main result announced in the abstract is equivalent to the following

Theorem 1.1 (Main). *For any \mathbb{A}^1 -invariant quasi-stable $\mathbb{Z}F_*$ -presheaf of abelian groups \mathcal{F} , the associated Nisnevich sheaf $\mathcal{F}_{\mathrm{nis}}$ is \mathbb{A}^1 -invariant provided that the base field is infinite. Moreover, if the base field k is perfect and infinite then every \mathbb{A}^1 -invariant quasi-stable Nisnevich framed sheaf is strictly \mathbb{A}^1 -invariant and quasi-stable.*

In the rest of the paper we suppose that *the base field k is infinite*.

Acknowledgements. The authors would like to thank Alexey Ananyevskiy, Andrey Druzhinin and Alexander Neshitov for many helpful discussions.

2. A FEW THEOREMS

The main aim of this section is to state a few major theorems on presheaves with framed transfers. As an application, we deduce the following result (which is the first assertion of Theorem 1.1).

Theorem 2.1. *For any \mathbb{A}^1 -invariant quasi-stable $\mathbb{Z}F_*$ -presheaf of abelian groups \mathcal{F} , the associated Nisnevich sheaf $\mathcal{F}_{\mathrm{nis}}$ is \mathbb{A}^1 -invariant and quasi-stable.*

We need some definitions. We will write $(V, \varphi; g)$ for an element a in $Fr_n(X, Y)$. We also write Z_a to denote the support of $(V, \varphi; g)$. It is a closed subset in $X \times \mathbb{A}^n$ which is finite over X and which coincides with the common vanishing locus of the functions $\varphi_1, \dots, \varphi_n$ in V . Next, by $\langle V, \varphi; g \rangle$ we denote the image of the element $1 \cdot (V, \varphi; g)$ in $\mathbb{Z}F_n(X, Y)$.

Definition 2.2. *Given any k -smooth variety X , there is a distinguished morphism $\sigma_X = (X \times \mathbb{A}^1, t, pr_X) \in Fr_1(X, X)$. Each morphism $f : Y \rightarrow X$ in Sm/k can be regarded tautologically as a morphism in $Fr_0(Y, X)$.*

In what follows by $SmOp/k$ we will mean a category whose objects are pairs (X, V) , where $X \in Sm/k$ and V is an open subset of X , with obvious morphisms of pairs.

Definition 2.3. Define $\mathbb{Z}F_*^{pr}(k)$ as an additive category whose objects are those of $SmOp/k$ and Hom-groups are defined as follows. We set for every $n \geq 0$ and $(X, V), (Y, W) \in SmOp/k$:

$$\mathbb{Z}F_*^{pr}((Y, W), (X, V)) = \ker[\mathbb{Z}F_n(Y, X) \oplus \mathbb{Z}F_n(W, V) \xrightarrow{i_Y^* - i_{X,*}} \mathbb{Z}F_n(W, X)],$$

where $i_Y : W \rightarrow Y$ is the embedding and $i_X : V \rightarrow X$ is the embedding. In other words, the group $\mathbb{Z}F_*^{pr}((Y, W), (X, V))$ consists of pairs $(a, b) \in \mathbb{Z}F_n(Y, X) \oplus \mathbb{Z}F_n(Y^0, X^0)$ such that $i_Y \circ b = a \circ i_X$. By definition, the composite $(a, b) \circ (a', b')$ is the pair $((a \circ b), (a' \circ b'))$.

We define $\overline{\mathbb{Z}F}_*(k)$ as an additive category whose objects are those of Sm/k and Hom-groups are defined as follows. We set for every $n \geq 0$ and $X, Y \in Sm/k$:

$$\overline{\mathbb{Z}F}_*(Y, X) = \text{Coker}[\mathbb{Z}F_*(\mathbb{A}^1 \times Y, X) \xrightarrow{i_0^* - i_1^*} \mathbb{Z}F_*(Y, X)].$$

Next, one defines $\overline{\mathbb{Z}F}_*^{pr}(k)$ as an additive category whose objects are those of $SmOp/k$ and Hom-groups are defined as follows. We set for every $n \geq 0$ and $(X, V), (Y, W) \in SmOp/k$:

$$\overline{\mathbb{Z}F}_*^{pr}((Y, W), (X, V)) = \text{Coker}[\mathbb{Z}F_*^{pr}(\mathbb{A}^1 \times (Y, W), (X, V)) \xrightarrow{i_0^* - i_1^*} \mathbb{Z}F_*^{pr}((Y, W), (X, V))].$$

Notation 2.4. Given $a \in \mathbb{Z}F_*(Y, X)$, denote by $[a]$ its class in $\overline{\mathbb{Z}F}_*(Y, X)$. Similarly, if $r = (a, b) \in \mathbb{Z}F_*^{pr}((Y, W), (X, V))$, then we will write $[[r]]$ to denote its class in $\overline{\mathbb{Z}F}_*^{pr}((Y, W), (X, V))$.

If X^0 is open in X and Y^0 is open in Y and $g(Z^0) \subset Y^0$ with Z^0 the support of $(V, \varphi; g)$, then $\langle\langle V, \varphi; g \rangle\rangle$ will stand for the element $(\langle V, \varphi; g \rangle, \langle V^0, \varphi^0; g^0 \rangle)$ in $\mathbb{Z}F_n((X, X^0), (Y, Y^0))$.

We will as well write $[V, \varphi; g]$ to denote the class of $\langle V, \varphi; g \rangle$ in $\overline{\mathbb{Z}F}_n(X, Y)$. In turn, $[[V, \varphi; g]]$ will stand for the class of $\langle\langle V, \varphi; g \rangle\rangle$ of $\overline{\mathbb{Z}F}_n((X, X^0), (Y, Y^0))$.

Remark 2.5. Clearly, the category $\mathbb{Z}F_*(k)$ is a full subcategory of $\mathbb{Z}F_*^{pr}(k)$ via the assignment $X \mapsto (X, \emptyset)$. Similarly, the category $\overline{\mathbb{Z}F}_*(k)$ is a full subcategory of $\overline{\mathbb{Z}F}_*^{pr}(k)$ via the assignment $X \mapsto (X, \emptyset)$.

In what follows we will also use the following category.

Definition 2.6. Let $\overline{\overline{\mathbb{Z}F}}_*(k)$ be a category whose objects are those of $SmOp/k$ and whose Hom-groups are obtained from the Hom-groups of the category $\overline{\mathbb{Z}F}_*^{pr}(k)$ by annihilating the identity morphisms $\text{id}_{(X, X)}$ of objects of the form (X, X) for all $X \in Sm/k$.

Notation 2.7. If $r = (a, b) \in \mathbb{Z}F_*^{pr}((Y, W), (X, V))$, then we will write $\overline{[[r]]}$ for its class in $\overline{\overline{\mathbb{Z}F}}_*(Y, W), (X, V)$. For (X, V) in $SmOp/k$ we write $\langle\langle \sigma_X \rangle\rangle$ for the morphism $(1 \cdot \sigma_X, 1 \cdot \sigma_V)$ in $\mathbb{Z}F_1((X, V), (X, V))$.

We will denote by $\overline{[[V, \varphi; g]]}$ the class of the element $[[V, \varphi; g]]$ in $\overline{\overline{\mathbb{Z}F}}_n((X, X^0), (Y, Y^0))$.

Construction 2.8. Let \mathcal{F} be an \mathbb{A}^1 -invariant $\mathbb{Z}F_*$ -presheaf of abelian groups. Then the assignments $(X, V) \mapsto \mathcal{F}(X, V) := \mathcal{F}(V)/\text{Im}(\mathcal{F}(X))$ and

$$(a, b) \mapsto [(a, b)^* = b^* : \mathcal{F}(V)/\text{Im}(\mathcal{F}(X)) \rightarrow \mathcal{F}(W)/\text{Im}(\mathcal{F}(Y))],$$

for any $(a, b) \in \mathbb{Z}F_*((Y, W), (X, V))$ define a presheaf \mathcal{F}^{pairs} on the category $\overline{\overline{\mathbb{Z}F}}_*(k)$.

The nearest aim is to formulate a series of theorems (each of which is of independent interest), which are crucial for the proof of Theorem 1.1.

Theorem 2.9 (Injectivity on affine line). *Let $U \subset \mathbb{A}_k^1$ be an open subset and let $i : V \hookrightarrow U$ be a non-empty open subset. Then there is a morphism $r \in \mathbb{Z}F_1(U, V)$ such that $[i] \circ [r] = [\sigma_U]$ in $\overline{\mathbb{Z}F_1}(U, U)$.*

Theorem 2.10 (Excision on affine line). *Let $U \subset \mathbb{A}_k^1$ be an embedding. Let $i : V \hookrightarrow U$ be an open inclusion with V non-empty. Let $S \subset V$ be a closed subset. Then there are morphisms $r \in \mathbb{Z}F_1((U, U - S), (V, V - S))$ and $l \in \mathbb{Z}F_1((U, U - S), (V, V - S))$ such that*

$$[[i]] \circ [[r]] = [[\sigma_U]] \text{ and } [[i]] \circ [[l]] = [[\sigma_V]]$$

in $\overline{\mathbb{Z}F_1}((U, U - S), (U, U - S))$ and $\overline{\mathbb{Z}F_1}((V, V - S), (V, V - S))$ respectively.

Theorem 2.11 (Injectivity for local schemes). *Let $X \in \text{Sm}/k$, $x \in X$ be a point, $U = \text{Spec}(\mathcal{O}_{X,x})$, $i : D \hookrightarrow X$ be a closed subset. Then there exists an integer N and a morphism $r \in \mathbb{Z}F_N(U, X - D)$ such that*

$$[r] \circ [j] = [\text{can}] \circ [\sigma_U^N]$$

in $\mathbb{Z}F_N(U, X)$ with $j : X - D \hookrightarrow X$ the open inclusion and $\text{can} : U \rightarrow X$ the canonical morphism.

Theorem 2.12 (Excision on relative affine line). *Let $X \in \text{Sm}/k$, $x \in X$ be a point, $W = \text{Spec}(\mathcal{O}_{X,x})$. Let $i : V = (\mathbb{A}_W^1)_f \subset \mathbb{A}_W^1$ be an affine open subset, where $f \in \mathcal{O}_{X,x}[t]$ is monic such that $f(0) \in \mathcal{O}_{X,x}^\times$. Then there are morphisms*

$r \in \mathbb{Z}F_1((\mathbb{A}_W^1, \mathbb{A}_W^1 - 0 \times W), (V, V - 0 \times W))$ and $l \in \mathbb{Z}F_1((\mathbb{A}_W^1, \mathbb{A}_W^1 - 0 \times W), (V, V - 0 \times W))$ such that

$$[[i]] \circ [[r]] = [[\sigma_{\mathbb{A}_W^1}]] \text{ and } [[l]] \circ [[i]] = [[\sigma_V]]$$

in $\overline{\mathbb{Z}F_1}((\mathbb{A}_W^1, \mathbb{A}_W^1 - 0 \times W), (\mathbb{A}_W^1, \mathbb{A}_W^1 - 0 \times W))$ and $\overline{\mathbb{Z}F_1}((V, V - 0 \times W), (V, V - 0 \times W))$ respectively.

To formulate further two theorems relating étale excision property, we need some preparations. Let $S \subset X$ and $S' \subset X'$ be closed subsets. Let

$$\begin{array}{ccc} V' & \longrightarrow & X' \\ \downarrow & & \downarrow \Pi \\ V & \longrightarrow & X \end{array}$$

be an elementary distinguished square with X and X' affine k -smooth. Let $S = X - V$ and $S' = X' - V'$ be closed subschemes equipped with reduced structures. Let $x \in S$ and $x' \in S'$ be two points such that $\Pi(x') = x$. Let $U = \text{Spec}(\mathcal{O}_{X,x})$ and $U' = \text{Spec}(\mathcal{O}_{X',x'})$. Let $\pi : U' \rightarrow U$ be the morphism induced by Π .

Theorem 2.13 (Injective étale excision). *Under the notation above there is an integer N and a morphism $r \in \mathbb{Z}F_N((U, U - S), (X', X' - S'))$ such that*

$$[[\Pi]] \circ [[r]] = [[\text{can}]] \circ [[\sigma_U^N]]$$

in $\overline{\mathbb{Z}F_N}((U, U - S), (X, X - S))$, where $\text{can} : U \rightarrow X$ is the canonical morphism.

Theorem 2.14 (Surjective étale excision). *Under the notation above suppose additionally that S is k -smooth. Then there are an integer N and a morphism $l \in \mathbb{Z}F_N((U, U - S), (X', X' - S'))$ such that*

$$[[l]] \circ [[\pi]] = [[\text{can}']] \circ [[\sigma_{U'}^N]]$$

in $\overline{\mathbb{Z}F_N}((U', U' - S'), (X', X' - S'))$ with $\text{can}' : U' \rightarrow X'$ the canonical morphism.

We are now in a position to prove the following

Theorem 2.15. *For any \mathbb{A}^1 -invariant quasi-stable $\mathbb{Z}F_*$ -presheaf of abelian groups \mathcal{F} the following statements are true:*

- (1) *under the assumptions of Theorem 2.9 the map $i^* : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is injective;*
- (2) *under the assumptions of Theorem 2.10 the map*

$$[[i]]^* : \mathcal{F}(U - S) / \mathcal{F}(U) \rightarrow \mathcal{F}(V - S) / \mathcal{F}(V)$$

is an isomorphism;

- (3) *under the assumptions of Theorem 2.11 the map*

$$\eta^* : \mathcal{F}(U) \rightarrow \mathcal{F}(\text{Spec}(k(X)))$$

is injective, where $\eta : \text{Spec}(k(X)) \rightarrow U$ is the canonical morphism;

- (3') *under the assumptions of Theorem 2.11 let U_x^h be the henselization of U at x and let $k(U_x^h)$ be the function field on U_x^h . Then the map*

$$\eta_h^* : \mathcal{F}(U_x^h) \rightarrow \mathcal{F}(\text{Spec}(k(U_x^h)))$$

is injective, where $\eta_h : \text{Spec}(k(U_x^h)) \rightarrow U_x^h$ is the canonical morphism;

- (4) *under the assumptions of Theorem 2.12 the map*

$$[[i]]^* : \mathcal{F}(\mathbb{A}_W^1 - 0 \times W) / \mathcal{F}(\mathbb{A}_W^1) \rightarrow \mathcal{F}(V - 0 \times W) / \mathcal{F}(V)$$

is an isomorphism;

- (5) *under the assumptions of Theorems 2.13 and 2.14 the map*

$$[[\Pi]]^* : \mathcal{F}(U - S) / \mathcal{F}(U) \rightarrow \mathcal{F}(U' - S') / \mathcal{F}(U')$$

is an isomorphism.

Proof. Firstly we may assume that \mathcal{F} is stable. Now assertions (1), (3) and (3') follow from Theorems 2.9 and 2.11. To prove assertions (2), (4) and (5), use Construction 2.8 and apply Theorems 2.10, 2.12, 2.13, and 2.14 respectively (recall that \mathcal{F} is stable). \square

Proof of Theorem 2.1. Firstly, (1) and (2) show that $\mathcal{F}|_{\mathbb{A}^1}$ is a Zariski sheaf. Using (5) applied to $X = \mathbb{A}^1$, one shows that for any open V in \mathbb{A}^1 one has $\mathcal{F}_{\text{Nis}}(V) = \mathcal{F}(V)$.

Now consider the following Cartesian square of schemes

$$\begin{array}{ccc} \text{Spec}(k(X)) & \xrightarrow{\eta} & X \\ i_{0,k(X)} \downarrow & & \downarrow i_{0,X} \\ \mathbb{A}_{k(X)}^1 & \xrightarrow{\eta \times id} & X \times \mathbb{A}^1 \end{array}$$

Evaluating the Nisnevich sheaf \mathcal{F}_{Nis} on this square, we get a square of abelian groups

$$\begin{array}{ccc} \mathcal{F}_{\text{Nis}}(\text{Spec}(k(X))) & \xleftarrow{\eta^*} & \mathcal{F}_{\text{Nis}}(X) \\ i_{0,k(X)}^* \uparrow & & \uparrow i_{0,X}^* \\ \mathcal{F}_{\text{Nis}}(\mathbb{A}_{k(X)}^1) & \xleftarrow{(\eta \times id)^*} & \mathcal{F}_{\text{Nis}}(X \times \mathbb{A}^1) \end{array}$$

The map $i_{0,X}^*$ is plainly surjective. It remains to check its injectivity. The map $(\eta \times id)^*$ is injective (apply (3')). As already mentioned in this proof, $\mathcal{F}_{\text{Nis}}(\mathbb{A}_{k(X)}^1) = \mathcal{F}(\mathbb{A}_{k(X)}^1)$. Since

$\mathcal{F}_{\text{Nis}}(\text{Spec}(k(X))) = \mathcal{F}(\text{Spec}(k(X)))$, we see that the map $i_{0,k(X)}^*$ is an isomorphism. Thus the map $i_{0,X}^*$ is injective. \square

We finish the section by proving the following useful statement, which is a consequence of Theorem 2.15(item 4):

Corollary 2.16. *Let $X \in \text{Sm}/k$, $x \in X$ be a point, $W = \text{Spec}(\mathcal{O}_{X,x})$. Let $\mathcal{V} := \text{Spec}(\mathcal{O}_{W \times \mathbb{A}^1, (x,0)})$ and $\text{can} : \mathcal{V} \hookrightarrow W \times \mathbb{A}^1$ be the canonical embedding. Let \mathcal{F} be an \mathbb{A}^1 -invariant stable $\mathbb{Z}F_*$ -presheaf of abelian groups. Then the pull-back map*

$$[[\text{can}]]^* : \mathcal{F}(W \times (\mathbb{A}^1 - \{0\})) / \mathcal{F}(W \times \mathbb{A}^1) \rightarrow \mathcal{F}(\mathcal{V} - W \times \{0\}) / \mathcal{F}(\mathcal{V})$$

is an isomorphism (both quotients make sense: the second quotient makes sense due to Theorem 2.15(item 3), the first one makes sense due to homotopy invariance of \mathcal{F}).

Proof. Consider the W -scheme $W \times \mathbb{P}^1$ and effective divisors of the form $H \sqcup D$ on $W \times \mathbb{P}^1$ such that H is a section of the projection $W \times \mathbb{P}^1 \rightarrow W$, D is a reduced divisor, $H \cap (W \times 0) = \emptyset$ and $D \cap (W \times 0) = \emptyset$. For such a divisor set $V_{H,D} := W \times \mathbb{P}^1 - (H \sqcup D)$. Note that $(W \times 0) \subset V_{H,D}$.

Consider the category, \mathcal{C} , of Zarisky neighborhoods of $(W \times 0)$ in $W \times \mathbb{P}^1$ as well as the presheaf $V \mapsto \mathcal{F}(V - W \times 0) / \text{Im}(\mathcal{F}(V))$ on \mathcal{C} . Clearly, the category \mathcal{C} is co-filtered. By definition, one has

$$\mathcal{F}(\mathcal{V}) = \varinjlim \mathcal{F}(V) \quad \text{and} \quad \mathcal{F}(\mathcal{V} - W \times 0) = \varinjlim \mathcal{F}(V - W \times 0),$$

where V runs over all Zarisky neighborhoods of $(W \times 0)$ in $W \times \mathbb{P}^1$. Thus

$$\mathcal{F}(\mathcal{V} - W \times \{0\}) / \mathcal{F}(\mathcal{V}) = \varinjlim \mathcal{F}(V - W \times 0) / \text{Im}(\mathcal{F}(V)).$$

Let \mathcal{C}' be the full subcategory of \mathcal{C} consisting of objects of the form $V_{H,D}$. Since the base field k is infinite and W is regular local, then the subcategory \mathcal{C}' is cofinal in \mathcal{C} . Thus,

$$\mathcal{F}(\mathcal{V} - W \times \{0\}) / \mathcal{F}(\mathcal{V}) = \varinjlim \mathcal{F}(V_{H,D} - W \times 0) / \text{Im}(\mathcal{F}(V_{H,D})).$$

We claim that for any inclusion $\varepsilon : V_{H_2,D_2} \hookrightarrow V_{H_1,D_1}$ the pull-back map

$$[[\varepsilon]]^* : \mathcal{F}(V_{H_1,D_1} - W \times 0) / \text{Im}(\mathcal{F}(V_{H_1,D_1})) \rightarrow \mathcal{F}(V_{H_2,D_2} - W \times 0) / \text{Im}(\mathcal{F}(V_{H_2,D_2}))$$

is an isomorphism. To prove this claim, note that the inclusion above yields an inclusion $H_1 \sqcup D_1 \subset H_2 \sqcup D_2$. We have that either $H_1 = H_2$ or $H_1 \subset D_2$. In the second case one has $H_1 \cap H_2 = \emptyset$, $H_1 \subset D_2$ and $H_2 \sqcup H_1 \subset H_2 \sqcup D_2$. If $H_1 = H_2$, then one has inclusions $V_{H_2,D_2} \xrightarrow{\alpha} V_{H_1,D_1} \xrightarrow{\beta} V_{H_1,0}$. Set $\gamma = \beta \circ \alpha$. By Theorem 2.15(item (4)) the maps $[[\beta]]^*$ and $[[\gamma]]^*$ are isomorphisms. Thus the map $[[\alpha]]^*$ is an isomorphism in this case, too. In the second case consider inclusions $V_{H_2,D_2} \xrightarrow{\gamma} V_{H_2,H_1} \xrightarrow{\beta} V_{H_2,0}$ and set $\alpha = \beta \circ \gamma$. By Theorem 2.15(item (4)) the maps $[[\beta]]^*$ and $[[\alpha]]^*$ are isomorphisms. Thus the map $[[\gamma]]^*$ is an isomorphism. Now consider inclusions $V_{H_2,H_1} \xrightarrow{\delta} V_{H_1,0}$ and $V_{H_1,D_1} \xrightarrow{\rho} V_{H_1,0}$. One has $\delta \circ \gamma = \rho \circ \varepsilon$. We already know that $[[\gamma]]^*$ is an isomorphism. By Theorem 2.15(item (4)) the maps $[[\rho]]^*$ and $[[\delta]]^*$ are isomorphisms. Thus $[[\varepsilon]]^*$ is an isomorphism in the second case, too. *The claim is proved.* Thus for any $V_{H,D} \in \mathcal{C}'$ the map

$$\mathcal{F}(V_{H,D} - W \times 0) / \text{Im}(\mathcal{F}(V_{H,D})) \rightarrow \mathcal{F}(\mathcal{V} - W \times \{0\}) / \mathcal{F}(\mathcal{V})$$

is an isomorphism. In particular, the map

$$\mathcal{F}(W \times \mathbb{A}^1 - W \times 0) / \mathcal{F}(W \times \mathbb{A}^1) = \mathcal{F}(V_{W \times \infty, 0} - W \times 0) / \mathcal{F}(V_{W \times \infty, 0}) \rightarrow \mathcal{F}(\mathcal{V} - W \times \{0\}) / \mathcal{F}(\mathcal{V})$$

is an isomorphism, whence the corollary. \square

3. NOTATION AND AGREEMENTS

Notation 3.1. Given a morphism $a \in Fr_n(Y, X)$, we will write $\langle a \rangle$ for the image of $1 \cdot a$ in $\mathbb{Z}F_n(Y, X)$ and write $[a]$ for the class of $\langle a \rangle$ in $\overline{\mathbb{Z}F_n}(Y, X)$.

Given a morphism $a \in Fr_n(Y, X)$, we will write Z_a for the support of a (it is a closed subset in $Y \times \mathbb{A}^n$ which finite over Y and determined by a uniquely). Also, we will often write

$$(\mathcal{V}_a, \varphi_a : \mathcal{V}_a \rightarrow Y \times \mathbb{A}^n; g_a : \mathcal{V}_a \rightarrow X)$$

for a representative of the morphism a (here $(\mathcal{V}_a, \rho : \mathcal{V}_a \rightarrow Y \times \mathbb{A}^n, s : Z_a \hookrightarrow \mathcal{V}_a)$ is an étale neighborhood of Z_a in $Y \times \mathbb{A}^n$).

Lemma 3.2. If the support Z_a of an element $a = (\mathcal{V}, \varphi; g) \in Fr_n(X, Y)$ is a disjoint union of Z_1 and Z_2 , then the element a determines two elements a_1 and a_2 in $Fr_n(X, Y)$. Namely, $a_1 = (\mathcal{V} - Z_2, \varphi|_{\mathcal{V}-Z_2}; g|_{\mathcal{V}-Z_2})$ and $a_2 = (\mathcal{V} - Z_1, \varphi|_{\mathcal{V}-Z_1}; g|_{\mathcal{V}-Z_1})$. Moreover, by the definition of $\mathbb{Z}F_n(X, Y)$ one has the equality

$$\langle a \rangle = \langle a_1 \rangle + \langle a_2 \rangle$$

in $\mathbb{Z}F_n(X, Y)$.

Definition 3.3. Let $i_Y : Y' \hookrightarrow Y$ and $i_X : X' \hookrightarrow X$ be open embeddings. Let $a \in Fr_n(Y, X)$. We say that the restriction $a|_{Y'}$ of a to Y' runs inside X' , if there is $a' \in Fr_n(Y', X')$ such that

$$i_X \circ a' = a \circ i_Y \quad (1)$$

in $Fr_n(Y', X')$.

It is easy to see that if there is a morphism a' satisfying condition (1), then it is unique. In this case the pair (a, a') is an element of $\mathbb{Z}F_n((Y, Y'), (X, X'))$. For brevity we will write $\langle\langle a \rangle\rangle$ for (a, a') .

Lemma 3.4. Let $i_Y : Y' \hookrightarrow Y$ and $i_X : X' \hookrightarrow X$ be open embeddings. Let $a \in Fr_n(Y, X)$. Let $Z_a \subset Y \times \mathbb{A}^n$ be the support of a . Set $Z'_a = Z_a \cap Y' \times \mathbb{A}^n$. Then the following are equivalent:

- (1) $g_a(Z'_a) \subset X'$;
- (2) the morphism $a|_{Y'}$ runs inside X' .

Proof. (1) \Rightarrow (2). Set $\mathcal{V}' = p_Y^{-1} \cap g^{-1}(X')$, where $p_Y = pr_Y \circ \rho_a : \mathcal{V} \rightarrow Y \times \mathbb{A}^n$. Then $a' := (\mathcal{V}', \varphi|_{\mathcal{V}'}; g|_{\mathcal{V}'}) \in Fr_n(Y', X')$ satisfies condition (1).

(2) \Rightarrow (1). If $a|_{Y'}$ runs inside X' , then for some $a' = (\mathcal{V}', \varphi'; g') \in Fr_n(Y', X')$ equality (1) holds. In this case the support Z' of a' must coincide with $Z'_a = Z_a \cap Y' \times \mathbb{A}^n$ and $g_a|_{Z'} = g'|_{Z'}$. Since $g'(Z')$ is a subset of X' , then $g_a(Z'_a) = g_a(Z') \subset X'$. \square

Corollary 3.5. Let $i_Y : Y' \hookrightarrow Y$ and $i_X : X' \hookrightarrow X$ be open embeddings. Let $h_\theta = (\mathcal{V}_\theta, \varphi_\theta; g_\theta) \in Fr_n(\mathbb{A}^1 \times Y, X)$. Suppose Z_θ , the support of h_θ , is such that for $Z'_\theta := Z_\theta \cap \mathbb{A}^1 \times Y' \times \mathbb{A}^n$ one has $g_\theta(Z'_\theta) \subset X'$. Then there are morphisms $\langle\langle h_\theta \rangle\rangle \in \mathbb{Z}F_n(\mathbb{A}^1 \times (Y, Y'), (X, X'))$, $\langle\langle h_0 \rangle\rangle \in \mathbb{Z}F_n((Y, Y'), (X, X'))$, $\langle\langle h_1 \rangle\rangle \in \mathbb{Z}F_n((Y, Y'), (X, X'))$ and one has an obvious equality

$$[[h_0]] = [[h_1]]$$

in $\overline{\mathbb{Z}F_n}((Y, Y'), (X, X'))$.

Lemma 3.6 (A disconnected support case). Let $i_Y : Y' \hookrightarrow Y$ and $i_X : X' \hookrightarrow X$ be open embeddings. Let $a \in Fr_n(Y, X)$ and let $Z_a \subset Y \times \mathbb{A}^n$ be the support of a . Set $Z'_a = Z_a \cap Y' \times \mathbb{A}^n$. Suppose that $Z_a = Z_{a,1} \sqcup Z_{a,2}$. For $i = 1, 2$ set $\mathcal{V}_i = \mathcal{V}_a - Z_{a,j}$ with $j \in \{1, 2\}$ and $j \neq i$. Also set $\varphi_i = \varphi_a|_{\mathcal{V}_i}$ and $g_i = g_a|_{\mathcal{V}_i}$. Suppose $a|_{Y'}$ runs inside X' , then

- (1) for each $i = 1, 2$ the morphism $a_i := (\mathcal{V}_i, \varphi_i; g_i)$ is such that $a_i|_{Y'}$ runs inside X' ;
(2) $\langle\langle a \rangle\rangle = \langle\langle a_1 \rangle\rangle + \langle\langle a_2 \rangle\rangle$ in $\mathbb{Z}F_n((Y, Y'), (X, X'))$.

4. SOME HOMOTOPIES

Suppose $U, W \subset \mathbb{A}_k^1$ are open and non-empty.

Lemma 4.1. *Let $a_0 = (\mathcal{V}, \varphi; g_0) \in Fr_1(U, W)$, $a_1 = (\mathcal{V}, \varphi; g_1) \in Fr_1(U, W)$. Suppose that the supports of a_0 and a_1 coincide. Denote their common support by Z . If $g_0|_Z = g_1|_Z$, then $[a_0] = [a_1]$ in $\mathbb{Z}F_1(U, W)$.*

Proof. Consider a function $g_\theta = (1 - \theta)g_0 + \theta g_1 : \mathbb{A}^1 \times \mathcal{V} \rightarrow \mathbb{A}^1$ and set $\mathcal{V}_\theta = g_\theta^{-1}(W)$, $\varphi_\theta = \varphi \circ pr_{\mathcal{V}} : \mathcal{V}_\theta \rightarrow \mathbb{A}_k^1$. Next, consider a homotopy

$$h_\theta = (\mathcal{V}_\theta, \varphi_\theta; g_\theta) \in Fr_1(\mathbb{A}^1 \times U, W). \quad (2)$$

The support of h_θ equals $\mathbb{A}^1 \times Z \subset \mathbb{A}^1 \times U \times \mathbb{A}^1$. Clearly, $h_0 = a_0$ and $h_1 = a_1$. Whence the lemma. \square

Corollary 4.2. *Under the assumptions of Lemma 4.1 let $U' \subset U$ and $W' \subset W$ be open subsets. Suppose that $a_0|_{U'}$ runs inside W' . Then $a_1|_{U'}$ runs inside W' , the restriction $h_\theta|_{\mathbb{A}^1 \times U'}$ of the homotopy h_θ runs inside W' and*

$$[[a_0]] = [[a_1]]$$

in $\overline{\mathbb{Z}F}_1((U, U'), (W, W'))$.

Lemma 4.3. *Let $a_0 = (\mathcal{V}, \varphi u_0; g) \in Fr_1(U, W)$, $a_1 = (\mathcal{V}, \varphi u_1; g) \in Fr_1(U, W)$, where $u_0, u_1 \in k[\mathcal{V}]$ are units. In this case the supports of a_0 and a_1 coincide. Denote their common support by Z . Suppose $u_0|_Z = u_1|_Z$, then $[a_0] = [a_1]$ in $\mathbb{Z}F_1(U, W)$.*

Proof. Set $u_\theta = (1 - \theta)u_0 + \theta u_1 \in k[\mathbb{A}^1 \times \mathcal{V}]$. Clearly, $u_\theta|_{\mathbb{A}^1 \times Z} = pr_Z^*(u_0) = pr_Z^*(u_1) \in k[\mathbb{A}^1 \times Z]$. Let $\mathcal{V}_\theta = \{u_\theta \neq 0\} \subset \mathbb{A}^1 \times \mathcal{V}$. Set,

$$h_\theta = (\mathcal{V}_\theta, u_\theta \varphi; g \circ pr_{\mathcal{V}}) \in Fr_1(\mathbb{A}^1 \times U, W). \quad (3)$$

The support of h_θ equals $\mathbb{A}^1 \times Z \subset \mathbb{A}^1 \times U \times \mathbb{A}^1$. Clearly, $h_0 = a_0$ and $h_1 = a_1$. Whence the lemma. \square

Corollary 4.4. *Under the assumptions of Lemma 4.3, let $U' \subset U$ and $W' \subset W$ be open subsets. Suppose $a_0|_{U'}$ runs inside W' . Then $a_1|_{U'}$ runs inside W' , the restriction $h_\theta|_{\mathbb{A}^1 \times U'}$ of the homotopy h_θ from the proof of Lemma 4.3 runs inside W' and*

$$[[a_0]] = [[a_1]]$$

in $\overline{\mathbb{Z}F}_1((U, U'), (W, W'))$.

Lemma 4.5. *Let $U \subset \mathbb{A}_k^1$ be non-empty open as above. Suppose $F_0(Y) = F_1(Y) \in k[U][Y]$. Let $\deg_Y(F_0) = \deg_Y(F_1) = d > 0$ and let their leading coefficients coincide and are units in $k[U]$. Then,*

$$[U \times \mathbb{A}^1, F_0(Y), pr_U] = [U \times \mathbb{A}^1, F_1(Y), pr_U] \in \mathbb{Z}F_1[U, U].$$

Proof. Set $F_\theta(Y) = (1 - \theta)F_0(Y) + \theta F_1(Y) \in k[U][\theta, Y]$. Consider a morphism

$$h_\theta = (\mathbb{A}^1 \times U \times \mathbb{A}^1, F_\theta; pr_{\mathbb{A}^1 \times U}) \in Fr_1(\mathbb{A}^1 \times U, U). \quad (4)$$

Clearly, $h_0 = (U \times \mathbb{A}^1, F_0(Y), pr_U)$ and $h_1 = (U \times \mathbb{A}^1, F_1(Y), pr_U)$. Whence the lemma. \square

Corollary 4.6. *Under the assumptions of Lemma 4.5 let $U' \subset U$ be an open subset. Then $(U \times \mathbb{A}^1, F_0(Y), pr_U)|_{U'}$, $(U \times \mathbb{A}^1, F_1(Y), pr_U)|_{U'}$ runs inside U' and the restriction $h_\theta|_{\mathbb{A}^1 \times U'}$ of the homotopy h_θ from the proof of Lemma 4.5 runs inside W' and*

$$[[U \times \mathbb{A}^1, F_0(Y), pr_U]] = [[U \times \mathbb{A}^1, F_0(Y), pr_U] \in \mathbb{Z}F_1((U, U'), (U, U'))]$$

in $\overline{\mathbb{Z}F}_1((U, U'), (U, U'))$.

Proposition 4.7. *Let $U \subset \mathbb{A}_k^1$ and $U' \subset U$ be open subsets. Let $t \in k[\mathbb{A}^1]$ be the standard parameter on \mathbb{A}_k^1 . Set $X := (t \otimes 1)|_{U \times U} \in k[U \times U]$ and $Y := (1 \otimes t)|_{U \times U} \in k[U \times U]$. Then for any integer $n \geq 1$, one has an equality*

$$[[U \times U, (Y - X)^{2n+1}, p_2]] = [[U \times U, (Y - X)^{2n}, p_2]] + [[\sigma_U]]$$

in $\overline{\mathbb{Z}F}_1((U, U'), (U, U'))$.

Proof. Let $m \geq 1$ be an integer. Then

$$[[U \times U, (Y - X)^m, p_2]] = [[U \times U, (Y - X)^m, p_1]] = [[U \times \mathbb{A}^1, (Y - X)^m, p_1]] = [[U \times \mathbb{A}^1, Y^m, p_1]] \quad (5)$$

in $\overline{\mathbb{Z}F}_1((U, U'), (U, U'))$. The first equality follows from Corollary 4.2, the third one follows from Corollary 4.6, the middle one is obvious.

There is a chain of equalities in $\overline{\mathbb{Z}F}_1((U, U'), (U, U'))$:

$$\begin{aligned} [[U \times \mathbb{A}^1, Y^{2n+1}, p_1]] &= [[U \times \mathbb{A}^1, Y^{2n}(Y + 1), p_1]] = \\ &= [[U \times (\mathbb{A}^1 - \{-1\}), Y^{2n}(Y + 1), p_1]] + [[U \times (\mathbb{A}^1 - \{0\}), Y^{2n}(Y + 1), p_1]] = \\ &= [[\mathcal{V}_0, Y^{2n}, p_1]] + [[\mathcal{V}_1, (Y + 1), p_1]] = \\ &= [[U \times \mathbb{A}^1, Y^{2n}, p_1]] + [[U \times \mathbb{A}^1, (Y + 1), p_1]]. \end{aligned}$$

Here the first equality holds by Corollary 4.6, the second one holds by Lemma 3.6, the third one holds by Corollary 4.4, the forth one is obvious (replacement of neighborhoods).

Continue the chain of equalities in $\overline{\mathbb{Z}F}_1((U, U'), (U, U'))$ as follows:

$$\begin{aligned} [[U \times \mathbb{A}^1, Y^{2n}, p_1]] + [[U \times \mathbb{A}^1, (Y + 1), p_1]] &= [[U \times \mathbb{A}^1, (Y - X)^{2n}, p_1]] + [[U \times \mathbb{A}^1, Y, p_1]] = \\ &= [[U \times \mathbb{A}^1, (Y - X)^{2n}, p_1]] + [[\sigma_U]] = [[U \times \mathbb{A}^1, (Y - X)^{2n}, p_2]] + [[\sigma_U]]. \end{aligned}$$

Here the first equality holds by Corollary 4.6, the second one holds by the definition of σ_U (see Notation 2.7), the third one holds by Corollary 4.2. We proved the equality

$$[[U \times \mathbb{A}^1, Y^{2n+1}, p_1]] = [[U \times \mathbb{A}^1, (Y - X)^{2n}, p_2]] + [[\sigma_U]]. \quad (6)$$

Combining that with the equality (5) for $m = 2n + 1$ we get the desired equality

$$[[U \times \mathbb{A}^1, Y^{2n+1}, p_2]] = [[U \times \mathbb{A}^1, (Y - X)^{2n}, p_2]] + [[\sigma_U]]$$

in $\overline{\mathbb{Z}F}_1((U, U'), (U, U'))$. Whence the proposition. \square

5. INJECTIVITY AND EXCISION ON AFFINE LINE

The aim of this section is to prove Theorems 2.9 and 2.10.

Lemma 5.1. *Let $U \subset \mathbb{A}^1$ be open and non-empty. Let $A = \mathbb{A}_k^1 - U$. Let $G_0(Y), G_1(Y) \in k[U][Y]$ be such that*

- (1) $\deg_Y(G_0) = \deg_Y(G_1)$;
- (2) *both are unitary in Y and the leading coefficients equal one;*
- (3) $G_0|_{U \times A} = G_1|_{U \times A} \in k[U \times A]^\times$.

Then

$$[U \times U, G_0; p_2] = [U \times U, G_1; p_2]$$

in $\overline{\mathbb{Z}F}_1(U, U)$.

Proof. One has a homotopy $h_\theta = (\mathbb{A}^1 \times U \times U, G_\theta, p_{\mathbb{A}^1 \times U}) \in Fr_1(\mathbb{A}^1 \times U, U)$, where $G_\theta = (1 - \theta)G_0 + \theta G_1$. Its restriction to $0 \times U$ and to $1 \times U$ coincides with morphisms $(U \times U, G_0; p_2)$ and $(U \times U, G_1; p_2)$ respectively. Whence the lemma. \square

Proof of Theorem 2.9. Under the assumptions of this theorem set $A = \mathbb{A}_k^1 - U$ and $B = U - V$. For each big enough integer $m \geq 0$ find a polynomial $F_m(Y) \in k[U][Y]$ such that $F_m(Y)$ is of degree m with the leading coefficient equal 1 and such that

- (i) $F_m(Y)|_{U \times A} = (Y - X)^m|_{U \times A} \in k[U \times A]^\times$;
- (ii) $F_m(Y)|_{U \times B} = 1 \in k[U \times B]^\times$.

Take $n \gg 0$ and set $r = \langle U \times V, F_{2n+1}; pr_V \rangle - \langle U \times V, F_{2n}; pr_V \rangle \in \mathbb{Z}F_1(U, V)$. Then one has a chain equalities in $\overline{\mathbb{Z}F}_1(U, U)$:

$$\begin{aligned} [i] \circ [r] &= [U \times U, F_{2n+1}; p_2] - [U \times U, F_{2n}; p_2] = [U \times U, (Y - X)^{2n+1}; p_2] - [U \times U, (Y - X)^{2n}; p_2] = \\ &= [\sigma_U]. \end{aligned}$$

Here the first equality is obvious, the second one holds by Lemma 5.1, the third one holds by Proposition 4.7. Whence the theorem. \square

Corollary 5.2 (of Lemma 5.1). *Under the conditions and notation of Lemma 5.1 let $S \subset U$ be a closed subset. Additionally to the conditions (1) – (3) suppose that the following conditions hold:*

- (4) $G_0(Y)|_{U \times S} = G_1(Y)|_{U \times S}$,
- (5) $G_0(Y)|_{(U-S) \times S}$ is invertible.

Then one has an equality

$$[[U \times U, G_0; p_2]] = [[U \times U, G_1; p_2]]$$

in $\overline{\mathbb{Z}F}_1((U, U - S), (U, U - S))$.

Proof of the corollary. The support Z_θ of the homotopy h_θ from the proof of Lemma 5.1 coincides with the vanishing locus of the polynomial G_θ . Since $G_\theta|_{\mathbb{A}^1 \times (U-S) \times S}$ is invertible, then $G_\theta \cap \mathbb{A}^1 \times (U - S) \times S = \emptyset$. By Lemma the homotopy $h_\theta|_{\mathbb{A}^1 \times (U-S) \times U}$ runs inside $U - S$. Hence

$$[[U \times U, G_0; p_2]] = [[h_0]] = [[h_1]] = [[U \times U, G_1; p_2]]$$

in $\overline{\mathbb{Z}F}_1((U, U - S), (U, U - S))$. In fact, the second equality here holds by Corollary 3.5. The first and the third equalities hold since for $i = 1, 2$ one has $h_i = (U \times U, G_i; p_2)$ in $Fr_1(U, U)$. \square

Proof of Theorem 2.10. Firstly construct a morphism $r \in \mathbb{Z}F_1((U, U - S), (V, V - S))$ such that for its class $[[r]]$ in $\overline{\mathbb{Z}F}_1((U, U - S), (V, V - S))$ one has

$$[[i]] \circ [[r]] = [[\sigma_U]] \quad (7)$$

in $\overline{\mathbb{Z}F}_1((U, U - S), (U, U - S))$.

To this end set $A = \mathbb{A}_k^1 - U$, $B = U - V$. Recall that $S \subset V$ is a closed subset. Take any big enough integer $m \geq 1$ and find a unitary polynomial $F_m(Y)$ of degree m satisfying the following properties:

- (i) $F_m(Y)|_{U \times A} = (Y - X)^m|_{U \times A} \in k[U \times A]^\times$;
- (ii) $F_m(Y)|_{U \times B} = 1 \in k[U \times B]^\times$;
- (iii) $F_m(Y)|_{U \times S} = (Y - X)^m|_{U \times S} \in k[U \times S]$.

Note that $F_m(Y)|_{(U-S) \times S} \in k[(U-S) \times S]^\times$. Hence by Lemma 3.4 the morphism $(U \times V, F_m; pr_V) \in Fr_1(U, V)$ being restricted to $U - S$ runs inside $V - S$. Thus using Definition 3.3 we get a morphism

$$\langle\langle U \times V, F_m; pr_V \rangle\rangle \in \mathbb{Z}F_1((U, U - S), (V, V - S)).$$

For that morphism one has equalities

$$[[i]] \circ [[U \times V, F_m; pr_V]] = [[U \times U, F_m; p_2]] = [[U \times U, (Y - X)^m; p_2]]$$

in $\mathbb{Z}F_1((U, U - S), (U, U - S))$. Here the first equality is obvious, the second one follows from Corollary 5.2. Take a big enough integer n . Set

$$r = \langle\langle U \times V, F_{2n+1}; pr_V \rangle\rangle - \langle\langle U \times V, F_{2n}; pr_V \rangle\rangle \in \mathbb{Z}F_1((U, U - S), (V, V - S)).$$

We claim that $[[i]] \circ [[r]] = [[\sigma_U]]$ in $\overline{\mathbb{Z}F}_1((U, U - S), (U, U - S))$. In fact,

$$[[i]] \circ [[r]] = [[U \times U, (Y - X)^{2n+1}; p_2]] - [[U \times U, (Y - X)^{2n}; p_2]] = [[\sigma_U]].$$

The first equality proven a few lines above and the second one follow from Proposition 4.7. Whence equality (7) holds.

Now find morphisms $l \in \mathbb{Z}F_1((U, U - S), (V, V - S))$ and $g \in \mathbb{Z}F_1((V, V - S), (V - S, V - S))$ such that

$$[[l]] \circ [[i]] - [[j]] \circ [[g]] = [[\sigma_V]] \quad (8)$$

in $\overline{\mathbb{Z}F}_1(V, V - S), (V, V - S)$. Here $j : (V - S, V - S) \rightarrow (V, V - S)$ is the inclusion. Clearly, equality (8) yields $[[l]] \circ [[i]] = [[\sigma_V]] \in \overline{\mathbb{Z}F}_1(V, V - S), (V, V - S)$.

Set $A' = \mathbb{A}_k^1 - U$, $B = U - V$ and recall that $S \subset V$ is a closed subset. Take an integer m big enough and find a monic in Y polynomial $F_m(Y) \in k[U][Y]$ such that

- (i) $F_m(Y)|_{U \times A'} = (Y - X)|_{U \times A} \in k[U \times A']^\times$;
- (ii) $F_m(Y)|_{U \times B} = 1 \in k[U \times B]^\times$;
- (iii) $F_m(Y)|_{U \times S} = (Y - X)|_{U \times S} \in k[U \times S]$.

Note that $F_m(Y)|_{(U-S) \times S} \in k[(U-S) \times S]^\times$. Hence by Lemma 3.4 the morphism $(U \times V, F_m; pr_V) \in Fr_1(U, V)$ being restricted to $U - S$ runs inside $V - S$. Thus, using Definition 3.3, we get a morphism

$$l = \langle\langle U \times V, F_m; pr_V \rangle\rangle \in \mathbb{Z}F_1((U, U - S), (V, V - S)).$$

To construct the desired morphism g , find a monic in Y polynomial $E_{m-1} \in k[V][Y]$ of degree $m - 1$ such that

- (i') $E_{m-1}(Y)|_{V \times A} = 1|_{U \times A} \in k[V \times A']^\times$;
- (ii') $E_{m-1}(Y)|_{V \times B} = (Y - X)^{-1} \in k[V \times B]^\times$;
- (iii') $E_{m-1}(Y)|_{V \times S} = 1|_{V \times S} \in k[V \times S]$;

$$(iv') \quad E_{m-1}(Y)|_{\Delta(V)} = 1|_{\Delta(V)} \in k[\Delta(V)].$$

Let $G \subset V \times \mathbb{A}_k^1$ be a closed subset defined by $E_{m-1}(Y) = 0$. By conditions (i') – (iv') one has $G \subset V \times (V - S)$ and $G \cap \Delta(V) = \emptyset$. Set $g' = \langle V \times (V - S) - \Delta(V), (Y - X)E_{m-1}(Y); pr_{V-S} \rangle \in \mathbb{Z}F_1(V, V - S)$. Since $g'|_{V-S} \in \mathbb{Z}F_1(V - S, V - S)$, we get a morphism

$$g = (g', g'|_{V-S}) \in \mathbb{Z}F_1((V, V - S), (V - S, V - S)). \quad (9)$$

Claim 5.3. *Equality (8) holds for the morphisms l and g defined above.*

Note firstly that $l \circ \langle i \rangle = \langle \langle V \times V, F_m(Y)|_{V \times V}; pr_2 \rangle \rangle \in \mathbb{Z}F_1((V, V - S), (V, V - S))$. Applying Corollary 5.2 to the case $V \subset \mathbb{A}^1$, $S \subset V$ and $A := A' \cup B$, we get an equality

$$[[V \times V, F_m(Y)|_{V \times V}; pr_2]] = [[V \times V, (Y - X)E_{m-1}(Y); pr_2]]$$

in $\overline{\mathbb{Z}F_1}((V, V - S), (V, V - S))$. By Lemma 3.6 and the fact that $G \cap \Delta(V) = \emptyset$, one has

$$[[V \times V, (Y - X)E_{m-1}(Y); pr_2]] =$$

$$[[V \times V - G, E_{m-1}(Y - X); pr_2]] + [[V \times V - \Delta(V), (Y - X)E_{m-1}(Y); pr_2]] =$$

$$= [[V \times V - G, E_{m-1}(Y - X); pr_2]] + [[j]] \circ [[g]]$$

in $\overline{\mathbb{Z}F_1}((V, V - S), (V - S, V - S))$.

One has a chain of equalities

$$[[V \times V - G, E_{m-1}(Y - X); pr_2]] = [[V \times V - G, (Y - X); pr_2]] = [[V \times V, (Y - X); pr_2]] =$$

$$= [[V \times V, Y; pr_1]] = [[V \times \mathbb{A}^1, Y; pr_1]] = [[\sigma_V]].$$

The first equality holds by condition (iv') and Corollary 4.4. The second one is obvious. The third one is equality (5) for $m = 1$ from the proof of Proposition 4.7. The forth one is the definition of $\langle \langle \sigma_V \rangle \rangle$ (see Definition 2.2 and Notation 2.7). Combining altogether, we get a chain of equalities

$$[[l]] \circ [[i]] = [[V \times V, F_m(Y)|_{V \times V}; pr_2]] = [[V \times V, (Y - X)E_{m-1}(Y); pr_2]] = [[\sigma_V]] + [[j]] \circ [[g]],$$

which proves the claim. Whence the theorem. \square

6. EXCISION ON RELATIVE AFFINE LINE

Proof of Theorem 2.12. Let $U = \mathbb{A}_W^1$, let $V \subset U$ be the open V from Theorem 2.12. Let $S = 0 \times W$. Note that $S \subset V$. Set $A = \mathbb{A}_W^1 - U = \emptyset$, $B = U - V = \{f = 0\}$. Then B is finite over U , since f is monic. Note that $B \cap (0 \times W) = \emptyset$.

Repeat literally the proof of Theorem 2.10 (see Section 5). \square

7. INJECTIVITY FOR LOCAL SCHEMES

The main aim of this section is to prove Theorem 2.11. Let $X \in Sm/k$, $x \in X$ be a point, $U = Spec(\mathcal{O}_{X,x})$, $i : D \hookrightarrow X$ be a closed subset. Under the notation of Theorem 2.11 we will construct an integer N and a morphism $r \in \mathbb{Z}F_N(U, X - D)$ such that

$$[r] \circ [j] = [can] \circ [\sigma_U^N]$$

in $\overline{\mathbb{Z}F_N}(U, X)$.

Let $X' \subset X$ be an open subset containing the point x and let $D' = X' \cap D$. Clearly, if we solve a similar problem for the triple U, X' and $X' - D'$, then we solve the problem for the original triple U, X and $X - D$. So, we may shrink X appropriately. In particular, we may assume that X is irreducible and the canonical sheaf $\omega_{X/k}$ is trivial, i.e. is isomorphic to the structure sheaf \mathcal{O}_X . Let $d = \dim X$.

Shrinking X more (and replacing D with its trace), we can find a commutative diagram of the form

$$\begin{array}{ccccc} \mathbb{A}^1 \times B & \xleftarrow{\pi} & X & \xleftarrow{i} & D \\ & \searrow p|_B & \downarrow p & \swarrow p|_D & \\ & & B & & \end{array} \quad (10)$$

where $p : X \rightarrow B$ is an almost elementary fibration in the sense of [4], B is an affine open subset of the projective space \mathbb{P}_k^{d-1} , π is a finite surjective morphism, $p|_D$ is a finite morphism.

The canonical sheaf $\omega_{X/k}$ remains to be trivial. Since p is an almost elementary fibration, then it is a smooth morphism such that for each point $b \in B$ the fibre $p^{-1}(b)$ is a $k(b)$ -smooth affine curve. Since π is finite, then the B -scheme X is affine.

Set $U = \text{Spec}(\mathcal{O}_{X,x})$, $\mathcal{X} = U \times_B X$, $\mathcal{D} = U \times_B D$. There is an obvious morphism $\Delta = (id, can) : U \rightarrow \mathcal{X}$. It is a section of the projection $p_U : \mathcal{X} \rightarrow U$. Let $p_X : \mathcal{X} \rightarrow X$ be the projection to X .

The base change of diagram (30) gives a commutative diagram of the form

$$\begin{array}{ccccc} \mathbb{A}^1 \times U & \xleftarrow{\Pi} & \mathcal{X} & \xleftarrow{i} & \mathcal{D} \\ & \searrow p|_U & \downarrow p_U & \swarrow p_U|_{\mathcal{D}} & \\ & & U & & \end{array} \quad (11)$$

Lemma 7.1 (Ojanguren-Panin). *There is a finite surjective morphism $H_\theta = (p_U, h_\theta) : \mathcal{X} \rightarrow \mathbb{A}^1 \times U$ of U -schemes such that for the closed subschemes $Z_1 := H_\theta^{-1}(1 \times U)$ and $Z_0 := H_\theta^{-1}(0 \times U)$ of \mathcal{X} one has*

- (i) $Z_1 \subset \mathcal{X} - \mathcal{D}$;
- (ii) $Z_0 = \Delta(U) \sqcup Z'_0$ (equality of schemes) and $Z'_0 \subset \mathcal{X} - \mathcal{D}$.

Now regard \mathcal{X} as an affine $\mathbb{A}^1 \times U$ -scheme via the morphism Π . And also regard \mathcal{X} as an X -scheme via p_X .

Remark 7.2. By Lemma 7.1 the class $[\mathcal{O}_{\mathcal{X}}]$ of the structure sheaf of the subscheme \mathcal{X} defines a morphism in $\text{Kor}_0(\mathbb{A}^1 \times U, X)$ such that for $i = 0, 1$ one has $[\mathcal{O}_{\mathcal{X}}]|_{\{i\} \times U} = [\mathcal{O}_{Z_i}]$. Moreover, one has $[\mathcal{O}_{Z_0}] = [can] + [j] \circ [\mathcal{O}_{Z'_0}]$ and $[\mathcal{O}_{Z'_0}] \in \text{Kor}_0(U, X - S)$ and $[\mathcal{O}_{Z_1}] \in \text{Kor}_0(U, X - S)$. Thus

$$[j] \circ ([\mathcal{O}_{Z_1}] - [\mathcal{O}_{Z'_0}]) = [can] \in \overline{\text{Kor}}_0(U, X).$$

Below we lift these elements to the category $\mathbb{Z}F_*(k)$ and equalities to the category $\overline{\mathbb{Z}F}_*(k)$.

Lemma 7.3. *There are an integer $N \geq 0$, a closed embedding $\mathcal{X} \hookrightarrow \mathbb{A}^1 \times U \times \mathbb{A}^N$ of $\mathbb{A}^1 \times U$ -schemes, an étale affine neighborhood $(\mathcal{V}, \rho : \mathcal{V} \rightarrow \mathbb{A}^1 \times U \times \mathbb{A}^N, s : \mathcal{X} \hookrightarrow \mathcal{V})$ of \mathcal{X} in $\mathbb{A}^1 \times U \times \mathbb{A}^N$, functions $\varphi_1, \dots, \varphi_N \in k[\mathcal{V}]$ and a morphism $r : \mathcal{V} \rightarrow \mathcal{X}$ such that:*

- (i) the functions $\varphi_1, \dots, \varphi_N$ generate the ideal $I_{s(\mathcal{X})}$ in $k[\mathcal{V}]$ defining the closed subscheme $s(\mathcal{X})$ of \mathcal{V} ;
- (ii) $r \circ s = id_{s(\mathcal{X})}$;

(iii) the morphism r is a U -scheme morphism if \mathcal{V} is regarded as a U -scheme via the morphism $pr_U \circ \rho$ and \mathcal{X} is regarded as a U -scheme via the morphism p_U .

By Lemma 7.1, $\mathcal{D}'_0 = \Delta(U) \sqcup \mathcal{D}_0$. Set $\mathcal{V}_0 = \rho^{-1}(0 \times U \times \mathbb{A}^N)$ and let \mathcal{W} be the henselization of \mathcal{V}_0 in $s(\Delta(U))$ (which is the same as the henselization of $0 \times U \times \mathbb{A}^N$ in $\Delta(U)$).

Remark 7.4. By Lemma 7.3 the functions $\phi_1|_{\mathcal{W}}, \dots, \phi_N|_{\mathcal{W}}$ generate the ideal I defining the closed subscheme $s(\Delta(U))$ of the scheme \mathcal{W} . In particular, the family

$$\overline{(\phi_1|_{\mathcal{W}})}, \dots, \overline{(\phi_N|_{\mathcal{W}})} \in I/I^2$$

is a free basis of the $k[U]$ -module I/I^2 . Another free basis of the $k[U]$ -module I/I^2 is the family

$$\overline{(t_1 - \Delta^*(t_1))|_{\mathcal{W}}}, \dots, \overline{(t_1 - \Delta^*(t_N))|_{\mathcal{W}}} \in I/I^2.$$

Let $A \in GL_N(k[\mathcal{W}])$ be a unique matrix which converts the second free basis to the first one and let $J := \det(A)$ be its determinant. Replacing ϕ_1 by $J^{-1}\phi_1$, we may and will assume below in this section that $J = 1 \in k[\mathcal{W}]$. This is useful to apply Theorem ?? below.

Set $\mathcal{V}_1 = \rho^{-1}(1 \times U \times \mathbb{A}^N) \cap r^{-1}(\mathcal{X} - \mathcal{D})$. Then $s(\mathcal{D}_1) \subset \mathcal{V}_1$. In fact, $(r \circ s)(\mathcal{D}_1) = \mathcal{D}_1 \subset \mathcal{X} - \mathcal{D}$ and $\rho(\mathcal{D}_1) \subset 1 \times U \times \mathbb{A}^N$. Thus $\mathcal{V}_1 \neq \emptyset$.

Construction 7.5 (Étale neighborhood of \mathcal{D}_1). The morphism $\rho| : \rho^{-1}(1 \times U \times \mathbb{A}^N) \rightarrow 1 \times U \times \mathbb{A}^N$ is étale and the inclusion $i_1 : \mathcal{V}_1 \hookrightarrow \rho^{-1}(1 \times U \times \mathbb{A}^N)$ is open. Set $\rho_1 = (\rho|) \circ i_1$. Then the triple

$$(\mathcal{V}_1, \rho_1 : \mathcal{V}_1 \rightarrow 1 \times U \times \mathbb{A}^N, s_1 = s|_{\mathcal{D}_1} : \mathcal{D}_1 \rightarrow \mathcal{V}_1)$$

is an étale neighborhood of \mathcal{D}_1 in $1 \times U \times \mathbb{A}^N$. Let $r_1 = r|_{\mathcal{V}_1} : \mathcal{V}_1 \rightarrow \mathcal{X} - \mathcal{D}$.

Definition 7.6. Set $a_1 = (\mathcal{D}_1, \mathcal{V}_1, \phi_1|_{\mathcal{V}_1}, \dots, \phi_N|_{\mathcal{V}_1}; (p_X)|_{\mathcal{X}-\mathcal{D}}) \circ r_1 \in Fr_N(U, X - D)$.

Set $\mathcal{V}'_0 = \rho^{-1}(0 \times U \times \mathbb{A}^N) \cap r^{-1}(\mathcal{X} - \mathcal{D})$. Then $s(\mathcal{D}'_0) \subset \mathcal{V}'_0$. In fact, $(r \circ s)(\mathcal{D}'_0) = \mathcal{D}'_0 \subset \mathcal{X} - \mathcal{D}$ and $\rho(\mathcal{D}'_0) \subset 0 \times U \times \mathbb{A}^N$. Thus $\mathcal{V}'_0 \neq \emptyset$.

Construction 7.7. The morphism $\rho| : \rho^{-1}(0 \times U \times \mathbb{A}^N) \rightarrow 0 \times U \times \mathbb{A}^N$ is étale and the inclusion $i_0 : \mathcal{V}'_0 \hookrightarrow \rho^{-1}(0 \times U \times \mathbb{A}^N)$ is open. Set $\rho_0 = (\rho|) \circ i_0$. Then the triple

$$(\mathcal{V}'_0, \rho_0 : \mathcal{V}'_0 \rightarrow 0 \times U \times \mathbb{A}^N, s_0 = s|_{\mathcal{D}'_0} : \mathcal{D}'_0 \rightarrow \mathcal{V}'_0)$$

is an étale neighborhood of \mathcal{D}'_0 in $0 \times U \times \mathbb{A}^N$. Let $r_0 = r|_{\mathcal{V}'_0} : \mathcal{V}'_0 \rightarrow \mathcal{X} - \mathcal{D}$.

Definition 7.8. Set $a_0 = (\mathcal{D}'_0, \mathcal{V}'_0, \phi_1|_{\mathcal{V}'_0}, \dots, \phi_N|_{\mathcal{V}'_0}; (p_X)|_{\mathcal{X}-\mathcal{D}}) \circ r_0 \in Fr_N(U, X - D)$.

Definition 7.9. Set $r = \langle a_1 \rangle - \langle a_0 \rangle \in \mathbb{Z}F_N(U, X - D)$.

Claim 7.10. One has an equality $[j] \circ [r] = [can] \circ [\sigma_U^N] \in \overline{\mathbb{Z}F}_N(U, X)$.

In fact, take an element $h_\theta = (\mathcal{X}, \mathcal{V}, \phi_1, \dots, \phi_N; p_X \circ r) \in Fr_N(\mathbb{A}^1 \times U, X)$. By Lemma 7.1 the support of h_θ is the closed subset $\Delta \sqcup \mathcal{D}'_0$. Thus by Lemma 3.2 $\langle h_\theta \rangle$ is the sum of two summands. Namely,

$$\langle h_\theta \rangle = j \circ \langle a_0 \rangle + \langle \Delta(U), \mathcal{W}, \phi_1|_{\mathcal{W}}, \dots, \phi_N|_{\mathcal{W}}; p_X \circ (r|_{s(\Delta(U))}) \rangle$$

in $\mathbb{Z}F_N(U, X)$. By Remark 7.4 and Theorem 12.1 for the second summand one has

$$[\Delta(U), \mathcal{W}, \phi_1|_{\mathcal{W}}, \dots, \phi_N|_{\mathcal{W}}; p_X \circ (r|_{s(\Delta(U))})] = [p_X \circ r|_{s(\Delta(U))} \circ (s \circ \Delta)] \circ [\sigma_U^N] = [can] \circ [\sigma_U^N]$$

in $\overline{\mathbb{Z}F}_N(U, X)$. Clearly, $h_1 = j \circ a_1$ in $Fr_N(U, X)$. Thus one has a chain of equalities

$$[j] \circ [a_1] = [h_1] = [h_0] = [j] \circ [a_0] + [can] \circ [\sigma_U^N]$$

in $\overline{\mathbb{Z}F}_N(U, X)$. Whence the Claim. Whence Theorem 2.11.

8. PRELIMINARIES FOR THE INJECTIVE PART OF THE ÉTALE EXCISION

Let $S \subset X$ and $S' \subset X'$ be closed subsets. Let

$$\begin{array}{ccc} V' & \longrightarrow & X' \\ \downarrow & & \downarrow \Pi \\ V & \longrightarrow & X \end{array}$$

be an elementary distinguished square with affine k -smooth X and X' . Let $S = X - V$ and $S' = X' - V'$ be closed subschemes equipped with reduced structures. Let $x \in S$ and $x' \in S'$ be two points such that $\Pi(x') = x$. Let $U = \text{Spec}(\mathcal{O}_{X,x})$ and $U' = \text{Spec}(\mathcal{O}_{X',x'})$. Let $\pi : U' \rightarrow U$ be the morphism induced by Π .

To prove Theorem 2.13, it suffices to find morphisms $a \in \mathbb{Z}F_N((U, U - S), (X', X' - S'))$ and $b_G \in \mathbb{Z}F_N((U, U - S), (X - S, X - S))$ such that

$$[[\Pi]] \circ [[a]] - [[j]] \circ [[b_G]] = [[can]] \circ [[\sigma_U^N]] \quad (12)$$

in $\overline{\mathbb{Z}F}_N(U, U - S), (X, X - S)$. Here $j : (X - S, X - S) \rightarrow (X, X - S)$ is the inclusion and $can : (U, U - S) \rightarrow (X, X - S)$ is the inclusion.

Let $in : X^\circ \hookrightarrow X$ and $in' : (X')^\circ \hookrightarrow X'$ be open such that

- (1) $x \in X^\circ$,
- (2) $x' \in (X')^\circ$,
- (3) $\Pi((X')^\circ) \subset X^\circ$,
- (4) the square

$$\begin{array}{ccc} V' \cap (X')^\circ & \longrightarrow & (X')^\circ \\ \downarrow & & \downarrow \Pi|_{(X')^\circ} \\ V \cap X^\circ & \longrightarrow & X^\circ \end{array}$$

is an elementary distinguished square.

Suppose morphisms $a^\circ \in \mathbb{Z}F_N((U, U - S), ((X')^\circ, (X')^\circ - S'))$, $b_G^\circ \in \mathbb{Z}F_N((U, U - S), (X^\circ - S, X^\circ - S))$ are such that for the inclusions $j^\circ : (X^\circ - S, X^\circ - S) \rightarrow (X^\circ, X^\circ - S)$ and $can_{X^\circ} : (U, U - S) \rightarrow (X^\circ, X^\circ - S)$ one has

$$[[\Pi|_{(X')^\circ}]] \circ [[a^\circ]] - [[j^\circ]] \circ [[b_G^\circ]] = [[can_{X^\circ}]] \circ [[\sigma_U^N]]. \quad (13)$$

Then the morphisms $a = in' \circ a^\circ$ and $b_G = in \circ b_G^\circ$ satisfy property (12). Thus if we shrink X and X' in such a way that properties (1) – (4) are fulfilled and find appropriate morphisms a^Y and b_G^Y , then we find a and b_G subjecting condition (12).

Remark 8.1. One way of shrinking X and X' such that properties (1) – (4) are fulfilled is as follows. Replace X by an affine open X° containing x and then replace X' by $(X')^\circ = \Pi^{-1}(X^\circ)$.

Let X'_n be the normalization of X in $\text{Spec}(k(X'))$. Let $\Pi_n : X'_n \rightarrow X$ be the corresponding finite morphism. Since X' is k -smooth it is an open subscheme of X'_n . Let $Y'' = X'_n - X'$. It is a closed subset in X'_n . Since $\Pi|_{S'} : S' \rightarrow S$ is a scheme isomorphism, then S' is closed in X'_n . Thus $S' \cap Y'' = \emptyset$. Hence there is a function $f \in k[X'_n]$ such that $f|_{Y''} = 0$ and $f|_{S'} = 1$.

Definition 8.2. Set $X'_{new} = (X'_n)_f$, $Y' = \{f = 0\}$, $Y = \Pi_n(Y'_{red}) \subset X$. Note that X'_{new} is an affine k -variety as a principal open subset of the affine k -variety X'_n . We regard Y' as an effective Cartier divisor of X'_n . The subset Y is closed in X , because Π_n is finite. Set $\Pi_{new} = \Pi|_{X'_{new}}$.

Remark 8.3. We have that $\Pi_{new}^{-1}(S) = S'$. Therefore the varieties X and X'_{new} are subject to properties (1) – (4) of the present section. Below we will work with this X'_{new} . However we will write X' for X'_{new} .

Lemma 8.4. Take X and X' as in Remark 8.3. Shrinking X and X' as described in Remark 8.1, one can find an almost elementary fibration $q : X \rightarrow B$ in the sense of [4] (B is affine open in \mathbb{P}^{n-1}) such that $q|_{Y \cup S}$ is finite, $\omega_{B/k} \cong \mathcal{O}_B$, $\omega_{X/k} \cong \mathcal{O}_X$.

The shrunk scheme X' will be regarded below as a B -scheme via the morphism $q \circ \Pi$.

Remark 8.5. If $q : X \rightarrow B$ is the almost elementary fibration from Lemma 8.4, then $\Omega_{X/B}^1 \cong \mathcal{O}_X$. In fact, $\omega_{X/k} \cong q^*(\omega_{B/k}) \otimes \omega_{X/B}$. Thus $\omega_{X/B} \cong \mathcal{O}_X$. Since X/B is a smooth relative curve, then $\Omega_{X/B}^1 = \omega_{X/B} \cong \mathcal{O}_X$.

If, furthermore, $j : X \hookrightarrow B \times \mathbb{A}^N$ is a closed embedding of B -schemes, then in $K_0(X)$ one has $[\mathcal{N}(j)] = (N-1)[\mathcal{O}_X]$, where $\mathcal{N}(j)$ is the normal bundle to X for the imbedding j .

Thus increasing the integer N , we may assume that the normal bundle $\mathcal{N}(j)$ is isomorphic to the trivial bundle \mathcal{O}_X^{N-1} .

Proposition 8.6. Let $q : X \rightarrow B$ be the almost elementary fibration from Lemma 8.4. Then there are an integer $N \geq 0$, a closed embedding $X \hookrightarrow B \times \mathbb{A}^N$ of B -schemes, an étale affine neighborhood $(\mathcal{V}, \rho : \mathcal{V} \rightarrow B \times \mathbb{A}^N, s : X \hookrightarrow \mathcal{V})$ of X in $B \times \mathbb{A}^N$, functions $\phi_1, \dots, \phi_{N-1} \in k[\mathcal{V}]$ and a morphism $r : \mathcal{V} \rightarrow X$ such that:

- (i) the functions $\phi_1, \dots, \phi_{N-1}$ generate the ideal $I_{s(X)}$ in $k[\mathcal{V}]$ defining the closed subscheme $s(X)$ of \mathcal{V} ;
- (ii) $r \circ s = id_X$;
- (iii) the morphism r is a B -scheme morphism if \mathcal{V} is regarded as a B -scheme via the morphism $pr_U \circ \rho$, and X is regarded as a B -scheme via the morphism q .

Definition 8.7. Let $x \in S, x' \in S'$ be such that $\Pi(x') = x$. Set $U = \text{Spec}(\mathcal{O}_{X,x})$. There is an obvious morphism $\Delta = (id, can) : U \rightarrow U \times_B X$. It is a section of the projection $p_U : U \times_B X \rightarrow U$. Let $p_X : U \times_B X \rightarrow X$ be the projection onto X . Let $\pi : U' \rightarrow U$ be the restriction of Π to U' .

Notation 8.8. In what follows we will write $U \times X$ to denote $U \times_B X$, $U \times X'$ to denote $U \times_B X'$, $U' \times X'$ to denote $U' \times_B X'$, etc. Here X' is regarded as a B -scheme via the morphism $q \circ \Pi$.

Proposition 8.9. Under the conditions of Lemma 8.4 and Notation 8.8 there is a function $h_\theta \in k[\mathbb{A}^1 \times U \times X]$ (θ is the parameter on the left factor \mathbb{A}^1) such that the following properties hold for the functions $h_\theta, h_1 := h_\theta|_{1 \times U \times X}$ and $h_0 := h_\theta|_{0 \times U \times X}$:

- (a) the morphism $(pr, h_\theta) : \mathbb{A}^1 \times U \times X \rightarrow \mathbb{A}^1 \times U \times \mathbb{A}^1$ is finite surjective, and hence the closed subscheme $Z_\theta := h_\theta^{-1}(0) \subset \mathbb{A}^1 \times U \times X$ is finite flat and surjective over $\mathbb{A}^1 \times U$;
- (b) for the closed subscheme $Z_0 := h_0^{-1}(0)$ one has $Z_0 = \Delta(U) \sqcup G$ (an equality of closed subschemes) and $G \subset U \times (X - S)$;
- (c) the closed subscheme $(id_U \times \Pi)^*(h_1) = 0$ is a disjoint union of the form $Z'_1 \sqcup Z'_2$ and $m := (id_U \times \Pi)|_{Z'_1}$ identifies Z'_1 with the closed subscheme $Z_1 := \{h_1 = 0\}$;
- (d) $Z_\theta \cap \mathbb{A}^1 \times (U - S) \times X = \emptyset$ or, equivalently, $Z_\theta \cap \mathbb{A}^1 \times (U - S) \times X \subset \mathbb{A}^1 \times (U - S) \times (X - S)$.

Remark 8.10. Item (d) yields the following inclusions: $Z_\theta \cap \mathbb{A}^1 \times (U - S) \times X \subset \mathbb{A}^1 \times (U - S) \times (X - S)$, $Z_0 \cap (U - S) \times X \subset (U - S) \times (X - S)$, and $Z_1 \cap (U - S) \times X \subset (U - S) \times (X - S)$. Applying item (c), we get another inclusion: $Z'_1 \cap (U - S) \times X' \subset (U - S) \times (X' - S')$.

Remark 8.11. The class $[\mathcal{O}_{Z_\theta}]$ of the structure sheaf of the subscheme Z_θ defines a morphism in $\text{Kor}_0(\mathbb{A}^1 \times (U, U - S), (X, X - S))$ such that for $i = 0, 1$ one has $[\mathcal{O}_{Z_\theta}]|_{\{i\} \times (U, U - S)} = [\mathcal{O}_{Z_i}]$. Moreover, by (b) one has $[\mathcal{O}_{Z_0}] = [\text{can}] + [j] \circ [\mathcal{O}_G]$ and $[\mathcal{O}_G] \in \text{Kor}_0((U, U - S), (X - S, X - S))$. Thus

$$[\Pi] \circ [\mathcal{O}_{Z_1}] = [\mathcal{O}_{Z_1}] = [\mathcal{O}_{Z_0}] = [\text{can}] + [j] \circ [\mathcal{O}_G] \in \overline{\text{Kor}}_0((U, U - S), (X, X - S)).$$

Below we lift these elements to the category $\mathbb{Z}F_*(k)$ and equalities to the category $\overline{\mathbb{Z}F}_*(k)$.

9. REDUCING THEOREM 2.13 TO PROPOSITIONS 8.6 AND 8.9

To construct a morphism $b \in \text{Fr}_N(U, X)$, we first construct its support in $U \times \mathbb{A}^N$ for an integer N , then we construct an étale neighborhood of the support in $U \times \mathbb{A}^N$, then one constructs a framing of the support in the neighborhood, and finally one constructs b itself. In the same manner we construct a morphism $a \in \text{Fr}_N(U, X')$ and a homotopy $H \in \text{Fr}_N(\mathbb{A}^1 \times U, X)$ between $\Pi \circ a$ and b . Using the fact that the support Z_0 of b is of the form $\Delta(U) \sqcup G$ with $G \subset U \times (X - S)$, we get an equality

$$\langle b \rangle = \langle b_1 \rangle + \langle b_2 \rangle$$

in $\mathbb{Z}F_N(U, X)$. Then we prove that $[b_1] = [\text{can}] \circ [\sigma_U^N]$ and $[b_2]$ factor through $X - S$. Moreover, we are able to work with morphisms of pairs. We will use systematically in this section the data from Proposition 8.6. The details are given below in this section.

Under the assumptions and notation of Proposition 8.6, Lemma 8.6 and Remark 8.3, set $\mathcal{V}' = X' \times_B V$. So we have a Cartesian square

$$\begin{array}{ccc} \mathcal{V}' & \xrightarrow{\Pi'} & \mathcal{V} \\ \downarrow r' & & \downarrow r \\ X' & \xrightarrow{\Pi} & X, \end{array}$$

where r' and Π' are the projections to the first and second factors respectively. The section $s : X \rightarrow \mathcal{V}$ defines a section $s' = (id, s) : X' \rightarrow \mathcal{V}'$ of r' . For brevity, we will write below $U \times \mathcal{V}$ to denote $U \times_B \mathcal{V}$, $U \times \mathcal{V}'$ for $U \times_B \mathcal{V}'$, and $id \times \rho$ for $id \times_B \rho : U \times_B \mathcal{V} \rightarrow U \times_B (B \times \mathbb{A}^n) = U \times \mathbb{A}^N$. Let $p_{\mathcal{V}} : U \times \mathcal{V} \rightarrow \mathcal{V}$ be the projection.

Let $X \subset B \times \mathbb{A}^N$ be the closed inclusion from Proposition 8.6. Taking the base change of the latter inclusion by means of the morphism $U \rightarrow B$, we get a closed inclusion $U \times X \subset U \times \mathbb{A}^N$.

Under the notation from Proposition 8.6 and Proposition 8.9, construct now a morphism $b \in \text{Fr}_N(U, X)$. Let $Z_0 \subset U \times X$ be the closed subset from Proposition 8.9. Then one has the closed inclusions

$$\Delta(U) \sqcup G = Z_0 \subset U \times X \subset U \times \mathbb{A}^N.$$

Let $in_0 : Z_0 \subset U \times X$ be the closed inclusion. Define an étale neighborhood of Z_0 in $U \times \mathbb{A}^N$ as follows:

$$(U \times \mathcal{V}, id \times \rho : U \times \mathcal{V} \rightarrow U \times \mathbb{A}^N, (id \times s) \circ in_0 : Z_0 \rightarrow U \times \mathcal{V}). \quad (14)$$

We will write $\Delta(U) \sqcup G = Z_0 \subset U \times \mathcal{V}$ for $((id \times s) \circ in_0)(Z_0) \subset U \times \mathcal{V}$. Let $f \in k[U \times \mathcal{V}]$ be a function such that $f|_G = 1$ and $f|_{\Delta(U)} = 0$. Then $\Delta(U)$ is a closed subset of the affine scheme $(U \times \mathcal{V})_f$.

Definition 9.1. Under the notation from Proposition 8.6 and Proposition 8.9, set

$$b' = (Z_0, U \times \mathcal{V}, p_{\mathcal{V}}^*(\varphi_1), \dots, p_{\mathcal{V}}^*(\varphi_{N-1}), (id \times r)^*(h_0); pr_X \circ (id \times r)) \in \text{Fr}_N(U, X).$$

We will sometimes write below $(Z_0, U \times \mathcal{V}, p_{\mathcal{V}}^*(\varphi), (id \times r)^*(h_0); pr_X \circ (id \times r))$ to denote the morphism b' .

To construct the desired morphism $b \in Fr_N(U, X)$, we need to modify slightly the function $p_{\mathcal{V}}^*(\varphi_1)$ in the framing of Z_0 . By Proposition 8.6 and item (b) of Proposition 8.9, the functions

$$p_{\mathcal{V}}^*(\varphi_1), \dots, p_{\mathcal{V}}^*(\varphi_{N-1}), (id \times r)^*(h_0)$$

generate an ideal $I_{(id \times s)(\Delta(U))}$ in $k[(U \times \mathcal{V})_f]$ defining the closed subscheme $\Delta(U)$ of the scheme $(U \times \mathcal{V})_f$. Let $t_1, t_2, \dots, t_N \in k[U \times \mathbb{A}^N]$ be the coordinate functions. For any $i = 1, 2, \dots, N$, set $t'_i = t_i - (t_i|_{\Delta(U)}) \in k[U \times \mathbb{A}^N]$. Then the family

$$(t''_1, t''_2, \dots, t''_N) = (id \times \rho)^*(t'_1), (id \times \rho)^*(t'_2), \dots, (id \times \rho)^*(t'_N)$$

also generates the ideal $I = I_{(id \times s)(\Delta(U))}$ in $k[(U \times \mathcal{V})_f]$. This holds since (14) is an étale neighborhood of Z_0 in $U \times \mathbb{A}^N$. By Remark 8.5 the $k[U] = k[(id \times s)(\Delta(U))]$ -module I/I^2 is free of rank N . Thus the families $(\bar{t}'_1, \bar{t}'_2, \dots, \bar{t}'_N)$ and $(\overline{p_{\mathcal{V}}^*(\varphi_1)}, \dots, \overline{p_{\mathcal{V}}^*(\varphi_{N-1})}, \overline{(id \times r)^*(h_0)})$ are two free bases of the $k[(id \times s)(\Delta(U))]$ -module I/I^2 . Let $J \in k[U]^\times$ be the Jacobian of a unique matrix $A \in M_N(k[U])$ which transforms the first free basis to the second one. Set,

$$\varphi_1^{new} = q_U^*(J^{-1})\varphi_1 \in k[\mathcal{V}],$$

where $q_U = pr_U \circ (id \times \rho) : \mathcal{V} \rightarrow U$. Let $A^{new} \in M_N(k[U])$ be a unique matrix changing the first free basis to the basis

$$(\overline{p_{\mathcal{V}}^*(\varphi_1^{new})}, \dots, \overline{p_{\mathcal{V}}^*(\varphi_{N-1})}, \overline{(id \times r)^*(h_0)}).$$

Then the Jacobian J^{new} of A^{new} is equal to 1:

$$J^{new} = 1 \in k[U]. \quad (15)$$

We will write

$$(\psi_1, \psi_2, \dots, \psi_{N-1}) \text{ for } (p_{\mathcal{V}}^*(\varphi_1^{new}), \dots, p_{\mathcal{V}}^*(\varphi_{N-1})).$$

Definition 9.2. Under the notation from Proposition 8.6 and Proposition 8.9 set

$$b = (Z_0, U \times \mathcal{V}, \psi_1, \dots, \psi_{N-1}, (id \times r)^*(h_0); pr_X \circ (id \times r)) \in Fr_N(U, X).$$

For brevity, we will sometimes write

$$b = (Z_0, U \times \mathcal{V}, p_{\mathcal{V}}^*(\psi), (id \times r)^*(h_0); pr_X \circ (id \times r)).$$

Under the notation from Proposition 8.6 and Proposition 8.9 construct now a morphism $a \in Fr_N(U, X)$. Let $Z_1 \subset U \times X$ be the closed subset from Proposition 8.9. Then one has closed inclusions

$$Z_1 \subset U \times X \subset U \times \mathbb{A}^N.$$

Set $(U \times X')_\circ = (U \times X') - Z_2''$ and $(U \times \mathcal{V}')_\circ = (id \times r')^{-1}((U \times X')_\circ)$. Let $in_1 : Z_1 \subset U \times X$ and $in'_1 : Z'_1 \subset (U \times X')_\circ$ be closed inclusions. Set,

$$r_\circ = (id \times r')|_{(U \times \mathcal{V}')_\circ} : (U \times \mathcal{V}')_\circ \rightarrow (U \times X')_\circ.$$

Using the notation of Proposition 8.6, define an étale neighborhood of Z_1 in $U \times \mathbb{A}^N$ as follows:

$$((U \times \mathcal{V}')_\circ, (id \times \rho) \circ (id \times \Pi') : (U \times \mathcal{V}')_\circ \rightarrow U \times \mathbb{A}^N, (id \times s') \circ in'_1 \circ m^{-1} : Z_1 \rightarrow (U \times \mathcal{V}')_\circ). \quad (16)$$

Definition 9.3. Under the notation of Proposition 8.6 and Proposition 8.9 set,

$$a = (Z_1, (U \times \mathcal{V}')_{\circ}, (id \times \Pi')^*(\psi_1), \dots, (id \times \Pi')^*(\psi_{N-1}), r_{\circ}^*(id \times \Pi)^*(h_1); pr_{X'} \circ r_{\circ}) \in Fr_N(U, X').$$

For brevity, we will sometimes write

$$a = (Z_1, (U \times \mathcal{V}')_{\circ}, (id \times \Pi')^*(\psi), r_{\circ}^*(id \times \Pi)^*(h_1); pr_{X'} \circ r_{\circ}).$$

Under the notation of Proposition 8.6 and Proposition 8.9, let us construct now a morphism $H_{\theta} \in Fr_N(\mathbb{A}^1 \times U, X)$. Let $Z_{\theta} \subset \mathbb{A}^1 \times U \times X$ be the closed subset from Proposition 8.9. Then one has closed inclusions

$$Z_{\theta} \subset \mathbb{A}^1 \times U \times X \subset \mathbb{A}^1 \times U \times \mathbb{A}^N.$$

Let $in_{\theta} : Z_{\theta} \subset \mathbb{A}^1 \times U \times X$ be the closed inclusion. Define an étale neighborhood of Z_{θ} in $\mathbb{A}^1 \times U \times \mathbb{A}^N$ as follows:

$$(\mathbb{A}^1 \times U \times \mathcal{V}, id \times id \times \rho : \mathbb{A}^1 \times U \times \mathcal{V} \rightarrow \mathbb{A}^1 \times U \times \mathbb{A}^N, (id \times id \times s) \circ in_{\theta} : Z_{\theta} \rightarrow \mathbb{A}^1 \times U \times \mathcal{V}). \quad (17)$$

Definition 9.4. Under the notation of Propositions 8.6 and 8.9 we set

$$H_{\theta} = (Z_{\theta}, \mathbb{A}^1 \times U \times \mathcal{V}, \psi_1, \dots, \psi_{N-1}, (id \times id \times r)^*(h_{\theta}); pr_X \circ (id \times id \times r)) \in Fr_N(\mathbb{A}^1 \times U, X).$$

We will sometimes write below $(Z_{\theta}, \mathbb{A}^1 \times U \times \mathcal{V}, \psi, (id \times id \times r)^*(h_{\theta}); pr_X \circ (id \times id \times r))$ to denote the morphism H_{θ} .

Lemma 9.5. One has equalities $H_0 = b$, $H_1 = \Pi \circ a$ in $Fr_N(U, X)$.

Proof. The first equality is obvious. To check the second one, consider

$$H_1 = (Z_1, U \times \mathcal{V}, \psi, (id \times r)^*(h_1); pr_X \circ (id \times r)) \in Fr_N(U, X).$$

Here we use $(U \times \mathcal{V}, id \times \rho : U \times \mathcal{V} \rightarrow U \times \mathbb{A}^N, (id \times s) \circ in_1 : Z_1 \rightarrow U \times \mathcal{V})$ as an étale neighborhood of Z_1 in $U \times \mathbb{A}^N$. Take another étale neighborhood of Z_1 in $U \times \mathbb{A}^N$

$$((U \times \mathcal{V}')_{\circ}, (id \times \rho) \circ (id \times \Pi') : (U \times \mathcal{V}')_{\circ} \rightarrow U \times \mathbb{A}^N, (id \times s') \circ in'_1 \circ m^{-1} : Z_1 \rightarrow (U \times \mathcal{V}')_{\circ})$$

and the morphism $id \times \Pi' : (U \times \mathcal{V}')_{\circ} \rightarrow U \times \mathcal{V}$ regarded as a morphism of étale neighborhoods. Refining the étale neighborhood of Z_1 in the definition of H_1 by means of that morphism, we get a N -frame $H'_1 = H_1$, which has the form

$$(Z_1, (U \times \mathcal{V}')_{\circ}, (id \times \Pi')^*(\psi), (id \times \Pi')^*(id \times r)^*(h_1); pr_X \circ (id \times r) \circ (id \times \Pi')).$$

Note that

$$(id \times \Pi')^*(id \times r)^*(h_1) = r_{\circ}^*(id \times \Pi)^*(h_1) \quad \text{and} \quad pr_X \circ (id \times r) \circ (id \times \Pi') = \Pi \circ pr_{X'} \circ r_{\circ}.$$

Thus, $H_1 = H'_1 = \Pi \circ a$ in $Fr_N(U, X)$. \square

The following lemma follows from Lemma 3.4 and Remark 8.10.

Lemma 9.6. The morphisms $a|_{U-S}$, $b|_{U-S}$, $H_{\theta}|_{\mathbb{A}^1 \times (U-S)}$ and $\Pi|_{X'-S'}$ run inside $X' - S'$, $X - S$, $X - S$ and $X - S$ respectively.

By the preceding lemma the morphisms a , b , H_{θ} and Π define morphisms

$$\langle\langle a \rangle\rangle \in ZF_N((U, U - S), (X', X' - S')), \langle\langle b \rangle\rangle \in ZF_N((U, U - S), (X, X - S)),$$

$$\langle\langle H_{\theta} \rangle\rangle \in ZF_N(\mathbb{A}^1 \times (U, U - S), (X, X - S)), \langle\langle \Pi \rangle\rangle \in ZF_N((X', X' - S'), (X, X - S)).$$

(see Definition 3.3). Lemma 9.5 and Definition 3.3 yield equalities

$$\langle\langle \Pi \rangle\rangle \circ \langle\langle a \rangle\rangle = \langle\langle H_1 \rangle\rangle \quad \text{and} \quad \langle\langle H_0 \rangle\rangle = \langle\langle b \rangle\rangle$$

in $\mathbb{Z}F_N((U, U - S), (X, X - S))$.

Corollary 9.7. *One has an equality $[[\Pi]] \circ [[a]] = [[b]]$ in $\overline{\mathbb{Z}F}_N((U, U - S), (X, X - S))$.*

Proof of Corollary 9.7. In fact, by Corollary 3.5 one has a chain of equalities

$$[[\Pi]] \circ [[a]] = [[H_1]] = [[H_0]] = [[b]]$$

in $\overline{\mathbb{Z}F}_N((U, U - S), (X, X - S))$. □

Reducing Theorem 2.13 to Propositions 8.6 and 8.9. The support Z_0 of b is the disjoint union $\Delta(U) \sqcup G$. Thus, by Lemma 3.6 one has an equality

$$\langle\langle b \rangle\rangle = \langle\langle b_1 \rangle\rangle + \langle\langle b_2 \rangle\rangle$$

in $\mathbb{Z}F_N((U, U - S), (X, X - S))$, where

$$b_1 = (\Delta(U), (U \times \mathcal{V})_f, \psi_1, \dots, \psi_{N-1}, (id \times r)^*(h_0); pr_X \circ (id \times r)),$$

$$b_2 = (G, (U \times \mathcal{V} - \Delta(U), \psi_1, \dots, \psi_{N-1}, (id \times r)^*(h_0); pr_X \circ (id \times r)).$$

By Proposition 8.9 one has $G \subset U \times (X - S)$. Thus $b_2 = j \circ b_G$ for an obvious morphism $b_G \in Fr_N(U, X - S)$. Also,

$$\langle\langle b_2 \rangle\rangle = \langle\langle j \rangle\rangle \circ \langle\langle b_G \rangle\rangle \in \mathbb{Z}F_N((U, U - S), (X, X - S)),$$

where $j : (X - S, X - S) \hookrightarrow (X, X - S)$ is a natural inclusion. By the latter comments and Corollary 9.7 one gets an equality

$$[[\Pi]] \circ [[a]] - [[j]] \circ [[b_G]] = [[b_1]]$$

in $\overline{\mathbb{Z}F}_N((U, U - S), (X, X - S))$. To prove equality (12), and hence to prove Theorem 2.13, it remains to check that $[[b_1]] = [[can]] \circ [[\sigma_U^N]]$. Recall that one has equality (15). Thus the relation $[[b_1]] = [[can]] \circ [[\sigma_U^N]]$ holds by Theorem 12.1. This finishes the proof of Theorem 2.13. □

10. PRELIMINARIES FOR THE SURJECTIVE PART OF THE ÉTALE EXCISION

Let $S \subset X$ and $S' \subset X'$ be closed subsets. Let

$$\begin{array}{ccc} V' & \longrightarrow & X' \\ \downarrow & & \downarrow \Pi \\ V & \longrightarrow & X \end{array}$$

be an elementary distinguished square with affine k -smooth X and X' . Let $S = X - V$ and $S' = X' - V'$ be closed subschemes equipped with reduced structures. Let $x \in S$ and $x' \in S'$ be two points such that $\Pi(x') = x$. Let $U = Spec(\mathcal{O}_{X,x})$ and $U' = Spec(\mathcal{O}_{X',x'})$. Let $\pi : U' \rightarrow U$ be the morphism induced by Π .

To prove Theorem 2.14 it suffices to find morphisms $a \in \mathbb{Z}F_N((U, U - S), (X', X' - S'))$ and $b_G \in \mathbb{Z}F_N((U', U' - S'), (X' - S', X' - S'))$ such that

$$[[a]] \circ [[\pi]] - [[j]] \circ [[b_G]] = [[can']] \circ [[\sigma_{U'}^N]] \quad (18)$$

in $\overline{\mathbb{Z}F}_N((U', U' - S'), (X', X' - S'))$. Here $j : (X' - S', X' - S') \rightarrow (X', X' - S')$ and $can' : (U', U' - S') \rightarrow (X', X' - S')$ are inclusions.

Replace X by an affine open neighborhood $in : X^\circ \hookrightarrow X$ of the point x . Replace X' by $(X')^\circ := \Pi^{-1}(X^\circ)$ and write $in' : (X')^\circ \hookrightarrow X'$ for the inclusion. Replace V by $V \cap X^\circ$ and V' with $V' \cap$

$(X')^\circ$. Let $\text{can}'_\circ : U' \rightarrow (X')^\circ$ be the canonical inclusion. Let $j^\circ : ((X')^\circ - S', (X')^\circ - S') \rightarrow ((X')^\circ, (X')^\circ - S')$ be an inclusion of pairs. If we find

$a^\circ \in \mathbb{Z}F_N((U, U - S), ((X')^\circ, (X')^\circ - S'))$ and $b_G^\circ \in \mathbb{Z}F_N((U', U' - S'), ((X')^\circ - S', (X')^\circ - S'))$ such that

$$[[a^\circ]] \circ [[\pi]] - [[j^\circ]] \circ [[b_G^\circ]] = [[\text{can}'_\circ]] \circ [[\sigma_{U'}^N]], \quad (19)$$

then the morphisms $a = \text{in}' \circ a^\circ$ and $b_G = \text{in}' \circ b_G^\circ$ satisfy condition (18).

Let X'_n be the normalization of X in $\text{Spec}(k(X'))$. Let $\Pi_n : X'_n \rightarrow X$ be the corresponding finite morphism. Since X' is k -smooth it is an open subscheme of X'_n . Let $Y'' = X'_n - X'$. It is a closed subset in X'_n . Since $\Pi|_{S'} : S' \rightarrow S$ is a scheme isomorphism, then S' is closed in X'_n . Thus $S' \cap Y'' = \emptyset$. Hence there is a function $f \in k[X'_n]$ such that $f|_{Y''} = 0$ and $f|_{S'} = 1$.

In this section we use agreements and notation from Definition 8.2 and Remark 8.3.

Proposition 10.1. *Let $q : X \rightarrow B$ be the almost elementary fibration from Lemma 8.4 and let $X' = X'_{\text{new}}$ be as in the Remark 8.3. Then there are an integer $N \geq 0$, a closed embedding $j : X' \hookrightarrow B \times \mathbb{A}^N$ of B -schemes, an étale affine neighborhood $(\mathcal{V}'', \rho'' : \mathcal{V}'' \rightarrow B \times \mathbb{A}^N, s'' : X' \hookrightarrow \mathcal{V}'')$ of X' in $B \times \mathbb{A}^N$, functions $\phi'_1, \dots, \phi'_{N-1} \in k[\mathcal{V}'']$ and a morphism $r'' : \mathcal{V}'' \rightarrow X'$ such that*

- (i) *the functions $\phi'_1, \dots, \phi'_{N-1}$ generate the ideal $I_{s''(X')}$ in $k[\mathcal{V}'']$ defining the closed subscheme $s''(X')$ of \mathcal{V}'' ;*
- (ii) *$r'' \circ s'' = \text{id}_{X'}$;*
- (iii) *the morphism r'' is a B -scheme morphism if \mathcal{V}'' is regarded as a B -scheme via the morphism $\text{pr}_U \circ \rho''$ and X' is regarded as a B -scheme via the morphism $q \circ \Pi$ from Lemma 8.4.*

Remark 10.2. *If $q : X \rightarrow B$ is the almost elementary fibration from Lemma 8.4, then $\Omega_{X'/B}^1 \cong \mathcal{O}_X$. In fact, by Remark 8.5 $\Omega_{X/B}^1 = \omega_{X/B} \cong \mathcal{O}_X$. The morphism $\Pi : X' \rightarrow X$ is étale. Thus $\Omega_{X'/B}^1 \cong \mathcal{O}_{X'}$.*

If, furthermore, $j : X' \hookrightarrow B \times \mathbb{A}^N$ is a closed embedding of B -schemes, then one has $[\mathcal{N}(j)] = (N-1)[\mathcal{O}_X]$ in $K_0(X)$, where $\mathcal{N}(j)$ is the normal bundle to X' associated with the imbedding j .

Thus increasing the integer N , we may assume that the normal bundle $\mathcal{N}(j)$ is isomorphic to the trivial bundle $\mathcal{O}_{X'}^{N-1}$.

Definition 10.3. *Let $x \in S$, $x' \in S'$ be such that $\Pi(x') = x$. We put $U = \text{Spec}(\mathcal{O}_{X,x})$. There is an obvious morphism $\Delta' = (\text{id}, \text{can}) : U' \rightarrow U' \times_B X'$. It is a section of the projection $p_{U'} : U' \times_B X' \rightarrow U'$. Let $p_{X'} : U' \times_B X' \rightarrow X'$ be the projection onto X' . Let $\pi : U' \rightarrow U$ be the restriction of Π to U' .*

Notation 10.4. *We regard X as a B -scheme via the morphism q and regard X' as a B -scheme via the morphism $q \circ \Pi$. In what follows we write $U \times X'$ for $U \times_B X'$, $U' \times X'$ for $U' \times_B X'$, etc.*

Proposition 10.5. *Under the conditions of Lemma 8.4 and Notation 10.4 there are functions $F \in k[U \times X']$ and $h'_\theta \in k[\mathbb{A}^1 \times U' \times X']$ (θ is the parameter on the left factor \mathbb{A}^1) such that the following properties hold for the functions h'_θ , $h'_1 := h'_\theta|_{1 \times U' \times X'}$ and $h'_0 := h'_\theta|_{0 \times U' \times X'}$:*

- (a) *the morphism $(\text{pr}, h'_\theta) : \mathbb{A}^1 \times U' \times X' \rightarrow \mathbb{A}^1 \times U' \times \mathbb{A}^1$ is finite and surjective, hence the closed subscheme $Z'_\theta := (h'_\theta)^{-1}(0) \subset \mathbb{A}^1 \times U' \times X'$ is finite flat and surjective over $\mathbb{A}^1 \times U'$;*
- (b) *for the closed subscheme $Z'_0 := (h'_0)^{-1}(0)$ one has $Z'_0 = \Delta'(U') \sqcup G'$ (an equality of closed subschemes) and $G' \subset U' \times (X' - S')$;*

- (c) $h'_1 = (\pi \times id)^*(F)$ (we write Z'_1 to denote the closed subscheme $\{h'_1 = 0\}$);
- (d) $Z'_\theta \cap \mathbb{A}^1 \times (U' - S') \times S' = \emptyset$ or, equivalently, $Z'_\theta \cap \mathbb{A}^1 \times (U' - S') \times X' \subset \mathbb{A}^1 \times (U' - S') \times (X' - S')$;
- (e) the morphism $(pr_U, F) : U \times X' \rightarrow U \times \mathbb{A}^1$ is finite surjective, and hence the closed subscheme $Z_1 := F^{-1}(0) \subset U \times X'$ is finite flat and surjective over U ;
- (f) $Z_1 \cap (U - S) \times S' = \emptyset$ or, equivalently, $Z_1 \cap (U - S) \times X' \subset (U - S) \times (X' - S')$.

Remark 10.6. Item (d) yields the following inclusions:

$$Z'_\theta \cap \mathbb{A}^1 \times (U' - S') \times X' \subset \mathbb{A}^1 \times (U' - S') \times (X' - S'),$$

$$Z'_0 \cap (U' - S') \times X' \subset (U' - S') \times (X' - S'),$$

$$Z'_1 \cap (U' - S') \times X' \subset (U' - S') \times (X' - S').$$

Applying (f) we get another inclusion: $Z_1 \cap (U - S) \times X' \subset (U - S) \times (X' - S')$.

Remark 10.7. The class $[\mathcal{O}_{Z'_\theta}]$ of the structure sheaf of the subscheme Z'_θ defines a morphism in $Kor_0(\mathbb{A}^1 \times (U', U' - S'), (X', X' - S'))$ such that for $i = 0, 1$ one has $[\mathcal{O}_{Z'_\theta}]|_{\{i\} \times (U', U' - S')} = [\mathcal{O}_{Z'_i}]$. Moreover, by (b) one has

$$[\mathcal{O}_{Z'_0}] = [can'] + [j] \circ [\mathcal{O}_{G'}]$$

and $[\mathcal{O}_{G'}] \in Kor_0((U', U' - S'), (X' - S', X' - S'))$. Thus,

$$[\mathcal{O}_{Z_1}] \circ [\pi] = [\mathcal{O}_{Z'_1}] = [\mathcal{O}_{Z'_0}] = [can'] + [j] \circ [\mathcal{O}_{G'}] \in \overline{Kor}_0((U', U' - S'), (X', X' - S')).$$

Below we will lift these elements to the category $\mathbb{Z}F_*(k)$ and relations to the category $\overline{\mathbb{Z}F}_*(k)$.

11. REDUCING THEOREM 2.14 TO PROPOSITIONS 10.1 AND 10.5

We suppose in this section that $S \subset X$ is k -smooth. To construct a morphism $a \in Fr_N(U, X')$, we first construct its support in $U \times \mathbb{A}^N$ for an integer N , then we construct an étale neighborhood of the support in $U \times \mathbb{A}^N$, then one constructs a framing of the support in the neighborhood and finally one constructs a itself. In the same fashion we construct a morphism $b \in Fr_N(U', X')$ and a homotopy $H \in Fr_N(\mathbb{A}^1 \times U', X')$ between $a \circ \pi$ and b . Using the fact that the support Z'_0 of b is of the form $\Delta'(U') \sqcup G'$ with $G' \subset U' \times (X' - S')$, we get a relation

$$\langle b \rangle = \langle b_1 \rangle + \langle b_2 \rangle$$

in $\mathbb{Z}F_N(U', X')$. Then we prove that $[b_1] = [can'] \circ [\sigma_{U'}^N]$ and $[b_2]$ factors through $X' - S'$. Moreover, we are able to work with morphisms of pairs. In this section we will use systematically the data from Propositions 10.1 and 10.5 and Notation 10.4. Details are given below in this section.

Let $X' \subset B \times \mathbb{A}^N$ be the closed inclusion from Proposition 10.1. Taking the base change of the latter inclusion by means of the morphism $U \rightarrow B$, we get a closed inclusion $U \times X' \subset U \times_B (B \times \mathbb{A}^N) = U \times \mathbb{A}^N$.

Under the notation from Proposition 10.1 and Proposition 10.5, construct now a morphism $b \in Fr_N(U', X')$. Let $Z'_0 \subset U' \times X'$ be the closed subset from Proposition 10.5. Then one has closed inclusions

$$\Delta'(U') \sqcup G' = Z'_0 \subset U' \times X' \subset U' \times \mathbb{A}^N.$$

Let $in_0 : Z'_0 \subset U' \times X'$ be a closed inclusion. Define an étale neighborhood of Z'_0 in $U' \times \mathbb{A}^N$ as follows:

$$(U \times \mathcal{V}'', id \times \rho'' : U' \times \mathcal{V}'' \rightarrow U' \times \mathbb{A}^N, (id \times s) \circ in_0 : Z_0 \rightarrow U \times \mathcal{V}). \quad (20)$$

We will write $\Delta'(U') \sqcup G' = Z'_0 \subset U' \times \mathcal{V}''$ for $((id \times s'') \circ in_0)(Z'_0) \subset U' \times \mathcal{V}''$. Let $f \in k[U \times \mathcal{V}'']$ be a function such that $f|_{G'} = 1$ and $f|_{\Delta'(U')} = 0$. Then $\Delta'(U')$ is a closed subset of the affine scheme $(U' \times \mathcal{V}'')_f$.

Definition 11.1. Under the notation from Proposition 8.6 and Proposition 8.9 set

$$b' = (Z'_0, U' \times \mathcal{V}'', (\pi \times id)^*(p_{\mathcal{V}''}^*(\phi'_1), \dots, p_{\mathcal{V}''}^*(\phi'_{N-1})), (id \times r'')^*(h'_0); pr_{X'} \circ (id \times r'')) \in Fr_N(U', X').$$

Here $p_{\mathcal{V}''} : U \times \mathcal{V}'' \rightarrow \mathcal{V}''$ is the projection. We will sometimes write below $(Z'_0, U' \times \mathcal{V}'', (\pi \times id)^*(p_{\mathcal{V}''}^*(\phi'_1)), (id \times r'')^*(h'_0); pr_{X'} \circ (id \times r''))$ to denote the morphism b' .

To construct the desired morphism $b \in Fr_N(U', X')$, we need to modify a bit the function $p_{\mathcal{V}''}^*(\phi'_1)$ in the framing of Z'_0 . By Proposition 10.1 and item (b) of Proposition 10.5, the functions

$$(\pi \times id)^*(p_{\mathcal{V}''}^*(\phi'_1)), \dots, (\pi \times id)^*(p_{\mathcal{V}''}^*(\phi'_{N-1})), (id \times r'')^*(h'_0)$$

generate an ideal $I_{(id \times s'')(\Delta'(U'))}$ in $k[(U' \times \mathcal{V}'')_f]$ defining the closed subscheme $\Delta'(U')$ of the scheme $(U' \times \mathcal{V}'')_f$. Let $t_1, t_2, \dots, t_N \in k[U' \times \mathbb{A}^N]$ be the coordinate functions. For any $i = 1, 2, \dots, N$, set $t'_i = t_i - (t_i|_{\Delta'(U')}) \in k[U' \times \mathbb{A}^N]$. Then the family

$$(t''_1, t''_2, \dots, t''_N) = (id \times \rho'')^*(t'_1), (id \times \rho'')^*(t'_2), \dots, (id \times \rho'')^*(t'_N)$$

also generates the ideal $I = I_{(id \times s'')(\Delta'(U'))}$ in $k[(U' \times \mathcal{V}'')_f]$. This holds since (20) is an étale neighborhood of Z'_0 in $U \times \mathbb{A}^N$. By Remark 10.2 the $k[U'] = k[(id \times s'')(\Delta'(U'))]$ -module I/I^2 is free of rank N . Thus the families

$$(\bar{t}''_1, \bar{t}''_2, \dots, \bar{t}''_N) \text{ and } (\overline{p_{\mathcal{V}''}^*(\phi'_1)}, \dots, \overline{p_{\mathcal{V}''}^*(\phi'_{N-1})}, \overline{(id \times r'')^*(h'_0)})$$

are two free bases of the $k[(id \times s'')(\Delta'(U'))]$ -module I/I^2 . Let $J \in k[U']^\times$ be the Jacobian of a unique matrix $A \in M_N(k[U'])$ changing the first free basis to the second one. There is an element $\lambda \in k[U]$ such that $\lambda|_{S \cap U} = J|_{S' \cap U'}$ (we identify here $S' \cap U'$ with $S \cap U$ via the morphism $\pi|_{S' \cap U'}$). Clearly, $\lambda \in k[U]^\times$.

Set

$$(\phi'_1)^{new} = q_U^*(J^{-1})(\phi'_1) \in k[\mathcal{V}''],$$

where $q_U = pr_U \circ (id \times \rho'') : \mathcal{V}'' \rightarrow U$. Let $A^{new} \in M_N(k[U])$ be a unique matrix which transforms the first free basis to the basis

$$(\overline{p_{\mathcal{V}''}^*((\phi'_1)^{new})}, \dots, \overline{p_{\mathcal{V}''}^*(\phi'_{N-1})}, \overline{(id \times r'')^*(h'_0)}).$$

Then the Jacobian J^{new} of A^{new} has the property:

$$J^{new}|_{S' \cap U'} = 1 \in k[S' \cap U']. \quad (21)$$

We will write

$$(\psi_1, \psi_2, \dots, \psi_{N-1}) \text{ for } (p_{\mathcal{V}''}^*((\phi'_1)^{new}), \dots, p_{\mathcal{V}''}^*(\phi'_{N-1})).$$

Definition 11.2. Under the notation from Proposition 10.1 and Proposition 10.5 set

$$b = (Z'_0, U' \times \mathcal{V}'', (\pi \times id)^*(\psi_1), \dots, (\pi \times id)^*(\psi_{N-1}), (id \times r'')^*(h'_0); pr_{X'} \circ (id \times r'')) \in Fr_N(U', X').$$

We will often write for brevity

$$b = (Z'_0, U' \times \mathcal{V}'', (\pi \times id)^*(\psi), (id \times r'')^*(h_0); pr_{X'} \circ (id \times r'')).$$

Under the notation from Proposition 10.1 and Proposition 10.5 construct now a morphism $a \in Fr_N(U, X')$. Let $Z_1 \subset U \times X'$ be the closed subset from Proposition 10.5. Then one has closed inclusions

$$Z_1 \subset U \times X' \subset U \times \mathbb{A}^N.$$

Let $in_1 : Z_1 \subset U \times X$ be the closed inclusion. Define an étale neighborhood of Z_1 in $U \times \mathbb{A}^N$ as follows:

$$(U \times \mathcal{V}'', id \times \rho'' : U \times \mathcal{V}'' \rightarrow U \times \mathbb{A}^N, (id \times s) \circ in_1 : Z_1 \hookrightarrow U \times \mathcal{V}''). \quad (22)$$

Definition 11.3. Under the notation from Proposition 10.1 and Proposition 10.5 set

$$a = (Z_1, U \times \mathcal{V}'', \psi_1, \dots, \psi_{N-1}, (id \times r'')^*(F); pr_{X'} \circ (id \times r'')) \in Fr_N(U, X')$$

We will sometimes write $(Z_1, U \times \mathcal{V}'', \psi, (id \times r'')^*(F); pr_{X'} \circ (id \times r''))$ to denote a .

Under the notation from Proposition 10.1 and Proposition 10.5 construct now a morphism $H_\theta \in Fr_N(\mathbb{A}^1 \times U', X')$. Let $Z'_\theta \subset \mathbb{A}^1 \times U' \times X'$ be the closed subset from Proposition 10.5. Then one has closed inclusions

$$Z'_\theta \subset \mathbb{A}^1 \times U' \times X' \subset \mathbb{A}^1 \times U' \times \mathbb{A}^N.$$

Let $in_\theta : Z'_\theta \subset \mathbb{A}^1 \times U' \times X'$ be the closed inclusion. Define an étale neighborhood of Z'_θ in $\mathbb{A}^1 \times U' \times \mathbb{A}^N$ as follows:

$$(\mathbb{A}^1 \times U' \times \mathcal{V}'', id \times id \times \rho'' : \mathbb{A}^1 \times U' \times \mathcal{V}'' \rightarrow \mathbb{A}^1 \times U' \times \mathbb{A}^N, (id \times id \times s'') \circ in_\theta : Z'_\theta \hookrightarrow \mathbb{A}^1 \times U' \times \mathcal{V}''). \quad (23)$$

Definition 11.4. Under the notation from Proposition 10.1 and Proposition 10.5 set H_θ to be equal to

$$(Z'_\theta, \mathbb{A}^1 \times U' \times \mathcal{V}'', pr^*((\pi \times id)^*(\psi)), (id \times id \times r'')^*(h'_\theta); pr_{X'} \circ (id \times id \times r'')) \in Fr_N(\mathbb{A}^1 \times U', X'),$$

where $pr : \mathbb{A}^1 \times U' \times \mathcal{V}'' \rightarrow U' \times \mathcal{V}''$ is the projection.

Lemma 11.5. One has equalities $H_0 = b$, $H_1 = a \circ \pi$ in $Fr_N(U', X')$.

Proof. The first equality is obvious. Let us prove the second one. By Proposition 10.5 one has $h'_1 = (\pi \times id)^*(F)$. Thus one has a chain of equalities in $Fr_N(U', X')$:

$$\begin{aligned} a \circ \pi &= (Z_1, U \times \mathcal{V}'', (\pi \times id)^*(\psi), (\pi \times id)^*((id \times r'')^*(F)); pr_{X'} \circ (id \times r'') \circ (\pi \times id)) = \\ &= (Z_1, U \times \mathcal{V}'', (\pi \times id)^*(\psi), (id \times r'')^*((\pi \times id)^*(F)); pr_{X'} \circ (\pi \times id) \circ (id \times r'')) = \\ &= (Z_1, U \times \mathcal{V}'', (\pi \times id)^*(\psi), (id \times r'')^*(h'_1); pr_{X'} \circ (id \times r'')) = H_1. \end{aligned}$$

□

The following lemma follows from Lemma 3.4 and Remark 10.6.

Lemma 11.6. The morphisms $a|_{U-S}$, $b|_{U'-S'}$, $H_\theta|_{\mathbb{A}^1 \times (U'-S')}$ and $\pi|_{U'-S'}$ run inside $X' - S'$, $X' - S'$, $X' - S'$ and $U - S$ respectively.

By the preceding lemma the morphisms a , b , H_θ and π define morphisms

$$\langle\langle a \rangle\rangle \in ZF_N((U, U - S), (X', X' - S')), \langle\langle b \rangle\rangle \in ZF_N((U', U' - S'), (X', X' - S')),$$

$$\langle\langle H_\theta \rangle\rangle \in ZF_N(\mathbb{A}^1 \times (U', U' - S'), (X', X' - S')), \langle\langle \pi \rangle\rangle \in ZF_N((U', U' - S'), (U, U - S)).$$

(see Definition 3.3). Lemma 11.5 and Definition 3.3 yield relations

$$\langle\langle a \rangle\rangle \circ \langle\langle \pi \rangle\rangle = \langle\langle H_1 \rangle\rangle \text{ and } \langle\langle H_0 \rangle\rangle = \langle\langle b \rangle\rangle$$

in $ZF_N((U', U' - S'), (X', X' - S'))$.

Corollary 11.7. *There is a relation $[[a]] \circ [[\pi]] = [[b]]$ in $\overline{\mathbb{Z}F}_N((U', U' - S'), (X', X' - S'))$.*

Proof of Corollary 11.7. In fact, by Corollary 3.5 one has a chain of equalities

$$[[a]] \circ [[\pi]] = [[H_1]] = [[H_0]] = [[b]]$$

in $\overline{\mathbb{Z}F}_N((U', U' - S'), (X', X' - S'))$. \square

Reducing Theorem 2.14 to Propositions 10.1 and 10.5. The support Z_0 of b is the disjoint union $\Delta'(U') \sqcup G'$. Thus, by Lemma 3.6 one has,

$$\langle\langle b \rangle\rangle = \langle\langle b_1 \rangle\rangle + \langle\langle b_2 \rangle\rangle$$

in $\mathbb{Z}F_N((U', U' - S'), (X', X' - S'))$, where

$$b_1 = (\Delta'(U'), (U' \times \mathcal{V}'')_f, \psi_1, \dots, \psi_{N-1}, (id \times r'')^*(h'_0); pr_{X'} \circ (id \times r'')),$$

$$b_2 = (G', (U' \times \mathcal{V}'' - \Delta'(U'), \psi_1, \dots, \psi_{N-1}, (id \times r'')^*(h'_0); pr_{X'} \circ (id \times r'')).$$

By Proposition 10.5 one has $G' \subset U' \times (X' - S')$. Thus $b_2 = j \circ b_{G'}$ for an obvious morphism $b_{G'} \in Fr_N(U', X' - S')$. Also,

$$\langle\langle b_2 \rangle\rangle = \langle\langle j \rangle\rangle \circ \langle\langle b_{G'} \rangle\rangle \in \mathbb{Z}F_N((U', U' - S'), (X', X' - S')),$$

where $j : (X' - S', X' - S') \hookrightarrow (X', X' - S')$ is a natural inclusion. By the latter comments and Corollary 11.7 one gets,

$$[[a]] \circ [[\pi]] - [[j]] \circ [[b_{G'}]] = [[b_1]]$$

in $\overline{\mathbb{Z}F}_N((U', U' - S'), (X', X' - S'))$. To prove equality (18), and hence to prove Theorem 2.14, it remains to check that $[[b_1]] = [[can']] \circ [[\sigma_{U'}^N]]$.

Let $U'' = (U')_{S' \cap U'}^h$ be the henselization of U' at $S' \cap U'$ and let $\pi' : U'' \rightarrow U'$ be the structure morphism. Recall that $S' \cap U'$ is essentially k -smooth. Thus the pair $(U'', S' \cap U')$ is a henselian pair with an essentially k -smooth closed subscheme $S' \cap U'$. Recall that one has equality (21). Thus by Theorem 12.2 one has an equality $[[b_1]] \circ [[\pi']] = [[can']] \circ [[\pi']] \circ [[\sigma_{U''}^N]]$ in $\overline{\mathbb{Z}F}_N((U'', U'' - S''), (X', X' - S'))$. Since $\pi' \circ \sigma_{U''}^N = \sigma_{U'}^N \circ \pi'$ one has,

$$[[b_1]] \circ [[\pi']] = [[can']] \circ [[\sigma_{U'}^N]] \circ [[\pi']] \in \overline{\mathbb{Z}F}_N((U'', U'' - S''), (X', X' - S')).$$

Applying Theorem 2.13 to the morphism $\pi' : U'' \rightarrow U'$, we see that for an integer $M \geq 0$ one has an equality

$$[[b_1]] \circ [[\sigma_{U'}^M]] = [[can']] \circ [[\sigma_{U'}^{M+N}]] \in \overline{\mathbb{Z}F}_{M+N}((U', U' - S'), (X', X' - S')).$$

Thus,

$$[[a]] \circ [[\pi]] \circ [[\sigma_{U'}^M]] - [[j]] \circ [[b_{G'}]] \circ [[\sigma_{U'}^M]] = [[can']] \circ [[\sigma_{U'}^{M+N}]] \in \overline{\mathbb{Z}F}_{M+N}((U', U' - S'), (X', X' - S')).$$

Since $\pi \circ \sigma_{U'}^M = \sigma_{U'}^M \circ \pi$, then we have that

$$[[a]] \circ [[\sigma_{U'}^M]] \circ [[\pi]] - [[j]] \circ [[b_{G'}]] \circ [[\sigma_{U'}^M]] = [[can']] \circ [[\sigma_{U'}^{M+N}]] \in \overline{\mathbb{Z}F}_{M+N}((U', U' - S'), (X', X' - S')).$$

Set $a_{new} = a \circ \sigma_{U'}^M$, $b_{G'}^{new} = b_{G'} \circ \sigma_{U'}^M$, $N(new) = M + N$. With these in hand the following equality holds:

$$[[a_{new}]] \circ [[\pi]] - [[j]] \circ [[b_{G'}^{new}]] = [[can']] \circ [[\sigma_{U'}^{N(new)}]] \in \overline{\mathbb{Z}F}_{M+N}((U', U' - S'), (X', X' - S')).$$

The latter equality is of the form (18). Whence Theorem 2.14. \square

12. TWO USEFUL THEOREMS

Theorem 12.1. *Let W be an essentially k -smooth local k -scheme and let $N \geq 1$ be an integer. Let $s : W \rightarrow W \times \mathbb{A}^N$ be a section of the projection $pr_W : W \times \mathbb{A}^N \rightarrow W$. Let*

$$((W \times \mathbb{A}^N)_{s(W)}^h, \rho : (W \times \mathbb{A}^N)_{s(W)}^h \rightarrow W \times \mathbb{A}^N, s^h : W \rightarrow (W \times \mathbb{A}^N)_{s(W)}^h)$$

be the henselization of $W \times \mathbb{A}^N$ at $s(W)$ (particularly, $s = \rho \circ s^h$). Let X be a k -smooth scheme. Suppose

$$\alpha = (s(W), (W \times \mathbb{A}^N)_{s(W)}^h, \phi_1, \dots, \phi_N; g) \in Fr_N(W, X),$$

is a N -framed correspondence such that the functions (ϕ_1, \dots, ϕ_N) generate the ideal $I = I_{s(W)}$ of these functions in $k[(W \times \mathbb{A}^N)_{s(W)}^h]$, which vanish on the closed subset $s(W)$. Let $A \in M_N(k[W])$ be a unique matrix transforming the free basis $(\overline{t_1 - (t_1|_{s(W)})}, \dots, \overline{t_N - (t_N|_{s(W)})})$ of the free $k[W]$ -module I/I^2 to the free basis $(\bar{\phi}_1, \dots, \bar{\phi}_N)$ of the same $k[W]$ -module. Suppose that the determinant $J := \det(A) = 1 \in k[W]$. Then,

$$[\alpha] = [g \circ s^h] \circ [\sigma_W^N] \in \overline{ZF}_N(W, X). \quad (24)$$

If $W^\circ \subset W$ is Zariski open and $X^\circ \subset X$ is Zariski open and $g(s^h(W^\circ)) \subset X^\circ$, then

$$[[\alpha]] = [[g \circ s^h]] \circ [[\sigma_W^N]] \in \overline{ZF}_N((W, W^\circ), (X, X^\circ)). \quad (25)$$

Theorem 12.2. *Let W be an essentially k -smooth local k -scheme and $N \geq 1$ be an integer. Let $S \subset W$ be a closed subscheme (essentially k -smooth) such that the pair (W, S) is henselian. Let X be a k -smooth scheme. Let $s : W \rightarrow W \times \mathbb{A}^N$, $\alpha \in Fr_N(W, X)$, $A \in M_N(k[W])$, $J := \det(A) \in k[W]$, s^h be the same as in Theorem 12.1. Suppose that $J|_S = 1 \in k[S]$. Then,*

$$[\alpha] = [g \circ s^h] \circ [\sigma_W^N] \in \overline{ZF}_N(W, X). \quad (26)$$

If $W^\circ \subset W$ is Zariski open and $X^\circ \subset X$ is Zariski open and $g(s(W^\circ)) \subset X^\circ$, then

$$[[\alpha]] = [[g \circ s^h]] \circ [[\sigma_W^N]] \in \overline{ZF}_N((W, W^\circ), (X, X^\circ)). \quad (27)$$

To prove these two theorems, we need some technical lemmas.

Lemma 12.3. *Let W be a k -smooth affine scheme and let $\mathcal{W} := (W \times \mathbb{A}^N)_{W \times 0}^h$ be the henselization of $W \times \mathbb{A}^N$ at $W \times 0$. Let $\mathcal{W}_\theta := (\mathbb{A}^1 \times W \times \mathbb{A}^N)_{\mathbb{A}^1 \times W \times 0}^h$ be the henselization of $\mathbb{A}^1 \times W \times \mathbb{A}^N$ at $\mathbb{A}^1 \times W \times 0$. Then there is a morphism $H_\theta : \mathcal{W}_\theta \rightarrow \mathcal{W}$ such that:*

- (a) $H_1 : \mathcal{W} \rightarrow \mathcal{W}$ is the identity morphism;
 - (b) $H_0 : \mathcal{W} \rightarrow \mathcal{W}$ coincides with the composite morphism $\mathcal{W} \rightarrow W \times \mathbb{A}^N \xrightarrow{pr_W} W \xrightarrow{s_0} \mathcal{W}$.
- If $W^\circ \subset W$ is open, then set $\mathcal{W}^\circ := p_{W \times \mathbb{A}^N}^{-1}(W^\circ \times \mathbb{A}^N)$ and $\mathcal{W}_\theta^\circ := p_{\mathbb{A}^1 \times W \times \mathbb{A}^N}^{-1}(\mathbb{A}^1 \times W^\circ \times \mathbb{A}^N)$. In that case $H_\theta(\mathcal{W}_\theta^\circ) \subset \mathcal{W}^\circ$.*

Corollary 12.4 (of Lemma 12.3). *Let $h_\theta = (\mathbb{A}^1 \times W \times 0, \mathcal{W}_\theta, \psi; H_\theta) \in Fr_N(\mathbb{A}^1 \times W, \mathcal{W})$. Then one has:*

- (a) $h_1 = (W \times 0, \mathcal{W}, \psi; id_{\mathcal{W}}) \in Fr_N(\mathbb{A}^1 \times W, \mathcal{W})$;
 - (b) $h_0 = (W \times 0, \mathcal{W}, \psi; s_0 \circ p_W) = s_0 \circ (W \times 0, \mathcal{W}, \psi; p_W) \in Fr_N(\mathbb{A}^1 \times W, \mathcal{W})$, where p_W is the composite map $\mathcal{W} \rightarrow W \times \mathbb{A}^N \xrightarrow{pr_W} W$ and $s_0 : W \hookrightarrow \mathcal{W}$ is the canonical inclusion.
- Moreover, if $W^\circ \subset W$ is open, then $h_\theta|_{\mathbb{A}^1 \times \mathcal{W}^\circ}$ runs inside \mathcal{W}° .*

Lemma 12.5. Let $(W \times 0, \mathcal{W}, \psi; p_W) \in Fr_N(W, W)$, where $p_W : \mathcal{W} \rightarrow W$ be the morphism from Corollary 12.4. Let $A_\theta \in GL_N(k[W][\theta])$ be a matrix such that $A_0 = id$. Set $A := A_1 \in GL_N(k[W])$. Take a row $(\psi', \dots, \psi'_N) := (\psi_1, \dots, \psi_N) \cdot p_W^*(A)$ in $k[\mathcal{W}]$ and take a N -framed correspondence

$$h_\theta := (\mathbb{A}^1 \times W \times 0, \mathbb{A}^1 \times \mathcal{W}, \Psi_\theta, p_W \circ pr_{\mathcal{W}}) \in Fr_N(\mathbb{A}^1 \times W, W),$$

where $\Psi_\theta = (pr_{\mathcal{W}}^*(\psi_1), \dots, pr_{\mathcal{W}}^*(\psi_N)) \cdot p_W^*(A_\theta)$ is a row in $k[\mathbb{A}^1 \times \mathcal{W}]$. Then one has:

(a) $h_0 = (W \times 0, \mathcal{W}, \psi, p_W)$;

(b) $h_1 = (W \times 0, \mathcal{W}, \psi', p_W)$.

Moreover, for any open $W^\circ \subset W$ the N -framed correspondence $h_\theta|_{\mathbb{A}^1 \times W^\circ}$ runs inside W° .

Lemma 12.6. Let $(W \times 0, \mathcal{W}, \psi; p_W) \in Fr_N(W, W)$ be as in Lemma 12.5. Suppose the functions ψ_1, \dots, ψ_N generate the ideal $I \subset k[\mathcal{W}]$ consisting of all the functions vanishing on the closed subset $W \times 0$. Furthermore, suppose that for any $i = 1, \dots, N$ one has that $\bar{\psi}_i = \bar{t}_i$ in I/I^2 . Set $\psi_{\theta,i} = (1 - \theta)\psi + \theta t_i \in k[\mathbb{A}^1 \times \mathcal{W}]$. Set $\psi_\theta := (\psi_{\theta,1}, \dots, \psi_{\theta,N})$. Set

$$h_\theta := (\mathbb{A}^1 \times W \times 0, \mathbb{A}^1 \times \mathcal{W}, \psi_\theta, p_W \circ pr_{\mathcal{W}}) \in Fr_N(\mathbb{A}^1 \times W, W).$$

Then one has:

(a) $h_0 = (W \times 0, \mathcal{W}, \psi; p_W)$;

(b) $h_1 = (W \times 0, \mathcal{W}, t_1, \dots, t_N; p_W) = (W \times 0, W \times \mathbb{A}^N, t_1, \dots, t_N; pr_W) = \sigma_W^N$.

Moreover, for any open $W^\circ \subset W$, the N -framed correspondence $h_\theta|_{\mathbb{A}^1 \times W^\circ}$ runs inside W° .

Lemma 12.7. Let $\alpha = (s(W), (W \times \mathbb{A}^N)_{s(W)}^h, \phi_1, \dots, \phi_N; g) \in Fr_N(W, X)$ be a N -framed correspondence, where $s : W \rightarrow W \times \mathbb{A}^N$, $(W \times \mathbb{A}^N)_{s(W)}^h$ and $s^h : W \rightarrow (W \times \mathbb{A}^N)_{s(W)}^h$ be as in Theorem 12.1. Let $T_s : W \times \mathbb{A}^N \rightarrow W \times \mathbb{A}^N$ be a morphism taking a point (w, v) to the point $(w, v + s(w))$. Let $T_s^h : \mathcal{W} = (W \times \mathbb{A}^N)_{W \times 0}^h \rightarrow (W \times \mathbb{A}^N)_{s(W)}^h$ be the induced morphism. Then one has,

$$[\alpha] = [W \times 0, \mathcal{W}, \phi_1 \circ T_s^h, \dots, \phi_N \circ T_s^h; g \circ T_s^h] \in \overline{\mathbb{Z}F}_N(W, X).$$

Moreover, if $W^\circ \subset W$ is open and if $X^\circ \subset X$ is any open such that $g(s^h(W^\circ)) \subset X^\circ$, then one has,

$$[[\beta]] = [[W \times 0, \mathcal{W}, \phi_1 \circ T_s^h, \dots, \phi_N \circ T_s^h; g \circ T_s^h]] \in \overline{\mathbb{Z}F}_N((W, W^\circ), (X, X^\circ)).$$

Proof of Theorem 12.1. Let $\alpha \in Fr_N(W, X)$ be the N -framed correspondence from Theorem 12.1. By Lemma 12.7 one has an equality in $\overline{\mathbb{Z}F}_N(W, X)$

$$[\alpha] = [W \times 0, \mathcal{W}, \psi_1, \dots, \psi_N; g \circ T_s^h] = [g \circ T_s^h] \circ [W \times 0, \mathcal{W}, \psi_1, \dots, \psi_N; id_{\mathcal{W}}],$$

where $\psi_i = \phi_i \circ T_s^h$. By Corollary 12.4 one has an equality in $\overline{\mathbb{Z}F}_N(W, \mathcal{W})$:

$$[W \times 0, \mathcal{W}, \psi_1, \dots, \psi_N; id_{\mathcal{W}}] = [s_0] \circ [W \times 0, \mathcal{W}, \psi; p_W].$$

Thus, one has

$$[\alpha] = [g \circ T_s^h \circ s_0] \circ [W \times 0, \mathcal{W}, \psi; p_W] = [g \circ s^h] \circ [W \times 0, \mathcal{W}, \psi; p_W] \in \overline{\mathbb{Z}F}_N(W, X).$$

Clearly, the functions (ψ_1, \dots, ψ_N) generate the ideal $I_0 = I_{W \times 0}$ of these functions in $k[\mathcal{W}]$, which vanish on the closed subset $W \times 0$. Let $A' \in M_N(k[W])$ be a unique matrix, which transforms the free basis $(\bar{t}_1, \dots, \bar{t}_N)$ of the free $k[W]$ -module I_0/I_0^2 to the free basis $(\bar{\psi}_1, \dots, \bar{\psi}_N)$ of the same $k[W]$ -module. Clearly, $\det(A') = \det(A)$. Thus $\det(A') = 1 \in k[W]$. The ring $k[W]$ is local. Thus A' belongs to the group of elementary $N \times N$ matrices over $k[W]$. Hence there is a matrix $A_\theta \in M_N(k[W][\theta])$ such that $A_0 = id$ and $A_1 = (A')^{-1} \in GL_N(k[W])$. By Lemma 12.5 one has an equality

$$[W \times 0, \mathcal{W}, \psi, p_W] = [W \times 0, \mathcal{W}, \psi', p_W] \in \overline{\mathbb{Z}F}_N(W, W)$$

with the row ψ'_1, \dots, ψ'_N as in Lemma 12.5. By construction, for any $i = 1, \dots, N$ the function ψ'_i has the property: $\bar{\psi}'_i = \bar{t}_i$ in I_0/I_0^2 . By Lemma 12.6 one has an equality

$$[W \times 0, \mathcal{W}, \psi', p_W] = [\sigma_W^N] \in \overline{ZF}_N(W, W).$$

Thus, finally,

$$[\alpha] = [g \circ s^h] \circ [\sigma_W^N] \in \overline{ZF}_N(W, X).$$

If $W^\circ \subset W$ is Zariski open and $X^\circ \subset X$ is Zariski open and $g(s^h(W^\circ)) \subset X^\circ$, then the same arguments prove the relation

$$[[\alpha]] = [[g \circ s^h]] \circ [[\sigma_W^N]] \in \overline{ZF}_N((W, W^\circ), (X, X^\circ)). \quad (28)$$

Theorem 12.1 is proved. \square

Proof of Theorem 12.2. Repeating literally the proof of Theorem 12.1, one gets an equality

$$[[\alpha]] = [[g \circ s^h]] \circ [[\sigma_W^{N-1}]] \circ [[J \cdot t]] \in \overline{ZF}_N((W, W^\circ), (X, X^\circ)), \quad (29)$$

where $[[J \cdot t]]$ is the class of the element $\langle J \cdot t \rangle$ corresponding to the 1-framed correspondence

$$(W \times 0, W \times \mathbb{A}^1, J \cdot t; pr_W) \in Fr_1(W, W)$$

by means of Definition 3.3. It remains to prove that $[[J \cdot t]] = [[t]] \in \overline{ZF}_N((W, W^\circ), (W, W^\circ))$. Let $i : S \hookrightarrow W$ be the inclusion. Since (W, S) is an affine henselian pair with an essentially k -smooth S , then there is a morphism $r : W \rightarrow S$ such that $r \circ i = id_S$. Moreover, there is a morphism $H_\theta : \mathbb{A}^1 \times W \rightarrow W$ such that $H_1 = id_W$ and H_0 is the composite map $W \xrightarrow{r} S \xrightarrow{i} W$. Consider an 1-framed correspondence of the form

$$h_\theta = (\mathbb{A}^1 \times W \times 0, \mathbb{A}^1 \times W \times \mathbb{A}^1, m \circ (J \times id_{\mathbb{A}^1}) \circ (H \times id); pr_{\mathbb{A}^1 \times W}) \in Fr_1(\mathbb{A}^1 \times W, \mathbb{A}^1 \times W),$$

where $m : \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is the multiplication. Clearly, $h_1 = (W \times 0, W \times \mathbb{A}^1, J \cdot t; pr_W)$ and

$$h_0 = (W \times 0, W \times \mathbb{A}^1, r^*(i^*(J)) \cdot t; pr_W) = (W \times 0, W \times \mathbb{A}^1, t; pr_W).$$

Thus $[[J \cdot t]] = [[t]] \in \overline{ZF}_N((W, W^\circ), (W, W^\circ))$. Theorem 12.2 is proved. \square

13. CONSTRUCTION OF h'_θ , F AND h_θ FROM PROPOSITIONS 10.5 AND 8.9

In this section we recall a modification of a result of M. Artin from [1] concerning existence of nice neighborhoods. The following notion (see [4, Defn.2.1]) is a modification of that introduced by Artin in [1, Exp. XI, Déf. 3.1].

Definition 13.1. An almost elementary fibration over a scheme B is a morphism of schemes $p : X \rightarrow B$ which can be included in a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{j} & \overline{X} & \xleftarrow{i} & X_\infty \\ & \searrow q & \downarrow \bar{q} & \swarrow q_\infty & \\ & & B & & \end{array} \quad (30)$$

of morphisms satisfying the following conditions:

- (i) j is an open immersion dense at each fibre of \bar{q} , and $X = \overline{X} - X_\infty$;
- (ii) \bar{q} is smooth projective all of whose fibres are geometrically irreducible of dimension one;
- (iii) q_∞ is a finite flat morphism all of whose fibres are non-empty;

(iv) the morphism i is a closed embedding and the ideal sheaf $I_{X_\infty} \subset \mathcal{O}_{\bar{X}}$ defining the closed subscheme X_∞ in \bar{X} is locally principal.

Let X and X' be as in Lemma 8.4 and let $q : X \rightarrow B$ be the almost elementary fibration from Lemma 8.4. The composite morphism $X' \xrightarrow{\bar{\Pi}} X \xrightarrow{j} \bar{X}$ is quasi-finite. Let \bar{X}' be the normalization of \bar{X} in $\text{Spec}(k(X'))$. Let $\bar{\Pi} : \bar{X}' \rightarrow \bar{X}$ be the canonical morphism (it is finite and surjective). Then $(\bar{\Pi})^{-1}(X)$ coincides with the normalization of X in $\text{Spec}(k(X'))$. Let $f' := f|_{(\bar{\Pi})^{-1}(X)}$, where f is from Definition 8.2. Let $Y' = \{f' = 0\}$ be the closed subscheme of $(\bar{\Pi})^{-1}(X)$. The morphism $(q \circ (\bar{\Pi}|_{(\bar{\Pi})^{-1}(X)}))|_{Y'} : Y' \rightarrow B$ is finite, since $q|_Y : Y \rightarrow B$ is finite and $\bar{\Pi}$ is finite. Thus Y' is closed in \bar{X}' . Since Y' is in $(\bar{\Pi})^{-1}(X)$ it has the empty intersection with X_∞ . Hence

$$X' = \bar{X}' - ((\bar{\Pi})^{-1}(X_\infty) \sqcup Y').$$

Both $(\bar{\Pi})^{-1}(X_\infty)$ and Y' are Cartier divisors in \bar{X}' . The Cartier divisor $(\bar{\Pi})^{-1}(X_\infty)$ is ample. Thus the Cartier divisor $D' := (\bar{\Pi})^{-1}(X_\infty) \sqcup Y'$ is ample as well and $(q \circ \bar{\Pi})|_{D'} : D' \rightarrow B$ is finite.

Set $\bar{\Gamma} = \bar{X}' \xrightarrow{(\bar{\Pi}, id)} \bar{X} \times_B \bar{X}'$ (the graph of the morphism B -morphism $\bar{\Pi}$). The projection $\bar{X} \times_B \bar{X}' \rightarrow \bar{X}'$ is a smooth morphism, since \bar{q} is smooth. The morphism $(\bar{\Pi}, id)$ is a section of the projection. Hence $\bar{\Gamma}$ is a Cartier divisor in $\bar{X} \times_B \bar{X}'$.

Set $\Gamma = pr_{\bar{X}}^{-1}(U) \subset U \times_B \bar{X}'$. Then $\Gamma \subset U \times_B \bar{X}'$ is a Cartier divisor. The scheme U' is contained in Γ as an open subscheme via the inclusion (π, can') , where $can' : U' \rightarrow X'$ is the canonical morphism. The composite morphism $pr_U \circ (\pi, can') : U' \rightarrow U$ coincides with $\pi : U' \rightarrow U$. Thus $pr_{\bar{X}}|_{\Gamma} : \Gamma \rightarrow U$ is étale at the points of U' .

Lemma 13.2. Set $\Gamma' = U' \times_U \Gamma \subset U' \times_U U \times_B \bar{X}' = U' \times_B \bar{X}'$. Then $\Gamma' \subset U' \times_B \bar{X}'$ is a Cartier divisor. Moreover,

$$\Gamma' = \Delta'(U') \sqcup G'$$

and $G' \cap U' \times_B S' = \emptyset$.

Remark 13.3. It is easy to check that $\Gamma \cap U \times_B S' = \delta(S')$, where $\delta(s') = (\pi(s'), s')$.

Definition 13.4. Set $\mathcal{D}' = U \times_B D'$ and $\mathcal{D}'' = U' \times_U \mathcal{D}' = U' \times_B D'$. They are Cartier divisors on $U \times_B \bar{X}'$ and $U' \times_B \bar{X}'$ respectively. Both \mathcal{D}' and \mathcal{D}'' are finite over B .

Let $s_0 \in \Gamma(U \times_B \bar{X}', \mathcal{L}(\mathcal{D}'))$ be the canonical section of the invertible sheaf $\mathcal{L}(\mathcal{D}')$ (its vanishing locus is \mathcal{D}'). Let $s_\Gamma \in \Gamma(U \times_B \bar{X}', \mathcal{L}(\Gamma))$ be the canonical section of the invertible sheaf $\mathcal{L}(\Gamma)$ (its vanishing locus is Γ). Let $s_{\Delta'(U')} \in \Gamma(U' \times_B \bar{X}', \mathcal{L}(\Delta'(U')))$ be the canonical section of the invertible sheaf $\mathcal{L}(\Delta'(U'))$ (its vanishing locus is $\Delta'(U')$). Let $s_{G'} \in \Gamma(U' \times_B \bar{X}', \mathcal{L}(G'))$ be the canonical section of the invertible sheaf $\mathcal{L}(G')$ (its vanishing locus is G'). Choose an integer $n \gg 0$.

Construction 13.5. Find a section $s_1 \in \Gamma(U \times_B \bar{X}', \mathcal{L}(n\mathcal{D}'))$ such that:

- (1) $s_1|_{U \times_B S'} = r_1 \otimes (s_\Gamma|_{U \times_B S'})$, where $r_1 \in \Gamma(U \times_B S', \mathcal{L}(n\mathcal{D}' - \Gamma))|_{U \times_B S'}$ has no zeros;
- (2) $s_1|_{\mathcal{D}'}$ has no zeros.

Construction 13.6. Set $t_1 := (\pi \times id)^*(s_1) \in \Gamma(U' \times_B \bar{X}', \mathcal{L}(n\mathcal{D}''))$. The properties of t_1 :

- (1') $t_1|_{U' \times_B S'} = r'_1 \otimes (s_{\Delta'(U')}|_{U' \times_B S'}) \otimes (s_{G'}|_{U' \times_B S'}) \cdot \lambda$, where $\lambda \in k[U']^\times$ and $r'_1 = (\pi \times id)^*(r_1) \in \Gamma(U' \times_B \bar{X}', \mathcal{L}(n\mathcal{D}'' - \Gamma'))$;
- (2') $t_1|_{\mathcal{D}''}$ has no zeros.

Construction 13.7. Construct a section $t_0 \in \Gamma(U' \times_B \bar{X}', \mathcal{L}(n\mathcal{D}''))$ of the form $t_0 = t'_0 \otimes s_{\Delta'(U')}$, where $t'_0 \in \Gamma(U' \times_B \bar{X}', \mathcal{L}(n\mathcal{D}'' - \Delta'(U')))$ satisfies the following conditions:

- (1'') $t'_0|_{\mathcal{D}''} = (t_1|_{\mathcal{D}''}) \otimes (s_{\Delta'(U')}|_{\mathcal{D}''})^{-1}$;
- (2'') $t'_0|_{U' \times_B S'} = r'_1 \otimes (s_{G'}|_{U' \times_B S'}) \cdot \lambda$, where r'_1 and λ are from Construction 13.6.

Lemma 13.8. One has:

- (1''') $t_0|_{\mathcal{D}''} = t_1|_{\mathcal{D}''}$ and both sections have no zeros on \mathcal{D}'' ;
- (2''') $t_0|_{U' \times_B S'} = t_1|_{U' \times_B S'}$ and both sections have no zeros on $(U' - S') \times_B S'$.

Indeed, the first equality is obvious. The second one follows from the chain of equalities

$$t_0|_{U' \times_B S'} = (t'_0|_{U' \times_B S'}) \otimes (s_{\Delta'(U')}|_{U' \times_B S'}) = r'_1 \otimes (s_{G'}|_{U' \times_B S'}) \otimes (s_{\Delta'(U')}|_{U' \times_B S'}) \cdot \lambda = t_1|_{U' \times_B S'}.$$

Definition 13.9. Let $s'_0 := (\pi \times id)^*(s_0) \in \Gamma(U' \times_B \bar{X}', \mathcal{L}(\mathcal{D}''))$. Set,

$$h'_\theta = \frac{((1 - \theta)t_0 + \theta t_1)|_{\mathbb{A}^1 \times U' \times_B X'}}{(s'_0)^{\otimes n}|_{\mathbb{A}^1 \times U' \times_B X'}} \in k[\mathbb{A}^1 \times U' \times_B X'] \text{ and } F = \frac{s_1|_{U \times_B X'}}{(s_0)^{\otimes n}|_{U \times_B X'}} \in k[U \times_B X'].$$

Proof of Proposition 10.5. The functions h'_θ and F satisfy properties (a) – (f) from Proposition 10.5. \square

In the rest of the section under the hypotheses of Proposition 8.9 we will construct a function $h_\theta \in k[\mathbb{A}^1 \times U \times X]$.

Let X and X' be as in Lemma 8.4 and let $q : X \rightarrow B$ be the almost elementary fibration from Lemma 8.4. The composite morphism $X' \xrightarrow{\Pi} X \xrightarrow{j} \bar{X}$ is quasi-finite. Let \bar{X}' be the normalization of \bar{X} in $\text{Spec}(k(X'))$. Let $\bar{\Pi} : \bar{X}' \rightarrow \bar{X}$ be the canonical morphism (it is finite and surjective). Let $X_\infty \subset \bar{X}$ be the Cartier divisor from diagram (30). Set $X'_\infty := (\bar{\Pi})^{-1}(X_\infty)$ (scheme-theoretically). Then X'_∞ is a Cartier divisor on \bar{X}' . Set

$$E := U \times_B X_\infty \text{ and } E' := U \times_B X'_\infty.$$

These are Cartier divisors on $U \times_B \bar{X}$ and $U \times_B \bar{X}'$ respectively and $(id \times \bar{\Pi})^*(E) = E'$.

Choose an integer $n \gg 0$. Find a section $r_1 \in \Gamma(U \times_B S, \mathcal{L}(nE - \Delta(U)|_{U \times_B S}))$ which has no zeros. Let $s_{\Delta(U)} \in \Gamma(U \times_B \bar{X}, \mathcal{L}(\Delta(U)))$ be the canonical section of the invertible sheaf $\mathcal{L}(\Delta(U))$ (its vanishing locus is $\Delta(U)$).

Construction 13.10. Find a section $s'_1 \in \Gamma(U \times_B \bar{X}', \mathcal{L}(nE'))$ such that the following holds

- (1) the Cartier divisor $Z'_1 := \{s'_1 = 0\}$ has the following properties:
 - (1a) $Z'_1 \subset U \times_B X'$;
 - (1a') the Cartier divisor Z'_1 is finite and étale over U ;
 - (1b) the morphism $i = (id \times \bar{\Pi})|_{Z'_1} : Z'_1 \hookrightarrow U \times_B X$ is a closed embedding; denote by Z_1 the closed subscheme $i(Z'_1)$ of the scheme $U \times_B X$;
 - (1c) $(id \times \bar{\Pi})^{-1}(Z_1) = Z'_1 \sqcup Z'_2$ (a scheme equality);
- (2) $s'_1|_{U \times_B S'} = (id \times \bar{\Pi})^*(s_{\Delta(U)})|_{U \times_B S'} \otimes ((id \times \bar{\Pi})|_{U \times_B S'})^*(r_1)$.

Remark 13.11. Properties (1a), (1a') and (1b) yield property (1c).

Lemma 13.12. One has $Z'_1 \cap U \times_B S' = \emptyset$.

Note that the Cartier divisor Z_1 in $U \times_B \bar{X}$ is equivalent to the Cartier divisor dnE , where $d = [k(X') : k(X)]$. Let $s_1 \in \Gamma(U \times_B \bar{X}, \mathcal{L}(Z_1))$ be the canonical section (its vanishing locus is Z_1). By property (1c) from Construction 13.10 one has an equality

$$(id \times \bar{\Pi})^*(s_1) = (s'_1 \otimes s'_2) \cdot \mu, \tag{31}$$

where $\mu \in k[U]^\times$ and $s'_2 \in \Gamma(U \times_B \bar{X}', \mathcal{L}(Z'_2))$ is the canonical section of the line bundle $\mathcal{L}(Z'_2)$.

Definition 13.13. Set $t_1 = s_1 \in \Gamma(U \times_B \bar{X}, \mathcal{L}(Z_1)) = \Gamma(U \times_B \bar{X}, \mathcal{L}(dnE))$.

Construction 13.14. Construct a section $t_0 \in \Gamma(U \times_B \bar{X}, \mathcal{L}(dnE))$ of the form $t_0 = s_{\Delta(U)} \otimes t'_0$, where $t'_0 \in \Gamma(U \times_B \bar{X}, \mathcal{L}(dnE - \Delta(U)))$, where $s_{\Delta(U)} \in \Gamma(U \times_B \bar{X}, \mathcal{L}(\Delta(U)))$ is the canonical section (its vanishing locus is $\Delta(U)$) and t'_0 has the following properties:

- (1') $t'_0|_E = (t_1|_E) \otimes (s_{\Delta(U)}|_E)^{-1}$;
- (2') $((id \times \bar{\Pi})|_{U \times_B S'})^*(t'_0|_{U \times_B S}) = ((id \times \Pi)|_{U \times_B S'})^*(r_1) \otimes (s'_2|_{U \times_B S'}) \cdot (\mu|_{U \times_B S'})$, where r_1, s'_1 are from Construction 13.10 and $\mu \in k[U]^\times$ is defined just above (since $U \times_B S' \cong U \times_B S$ condition (2') on t'_0 is a condition on $t'_0|_{U \times_B S}$).

Lemma 13.15. The following statements are true:

- (1'') $t_0|_E = t_1|_E$ and both sections have no zeros on E ;
- (2'') $t_0|_{U \times_B S} = t_1|_{U \times_B S}$ and both sections have no zeros on $(U - S) \times_B S$.

Indeed, the first equality is obvious. To prove the second one, it suffices to prove the equality

$$((id \times \bar{\Pi})|_{U \times_B S'})^*(t_0|_{U \times_B S}) = ((id \times \bar{\Pi})|_{U \times_B S'})^*(t_1|_{U \times_B S}).$$

This equality is a consequence of the following chain of equalities:

$$\begin{aligned} ((id \times \bar{\Pi})|_{U \times_B S'})^*(t_0|_{U \times_B S}) &= (id \times \bar{\Pi})^*(s_{\Delta(U)})|_{U \times_B S'} \otimes ((id \times \Pi)|_{U \times_B S'})^*(r_1) \otimes (s'_2|_{U \times_B S'}) \cdot (\mu|_{U \times_B S'}) = \\ &= s'_1|_{U \times_B S'} \otimes (s'_2|_{U \times_B S'}) \cdot (\mu|_{U \times_B S'}) = ((id \times \bar{\Pi})|_{U \times_B S'})^*(t_1|_{U \times_B S}). \end{aligned}$$

The first equality holds by property (2') from Construction 13.14, the second equality holds by property (2) from Construction 13.10, the third one follows from equality (31) and Definition 13.13.

Definition 13.16. Set,

$$h_\theta = \frac{((1 - \theta)t_0 + \theta t_1)|_{\mathbb{A}^1 \times U \times X}}{(s_E^{\otimes dn})|_{\mathbb{A}^1 \times U \times X}} \in k[\mathbb{A}^1 \times U \times X].$$

Proof of Proposition 8.9. The function h_θ satisfies properties (a) – (d) from Proposition 8.9. \square

14. HOMOTOPY INVARIANCE OF COHOMOLOGY PRESHEAVES

In this section we prove Theorems 14.13 and 14.14. They complete the proof of Theorem 1.1, which is the main result of the paper.

Definition 14.1. Let \mathcal{G} be a homotopy invariant presheaf of abelian groups with $\mathbb{Z}F_*$ -transfers. Then the presheaf $X \mapsto \mathcal{G}_{-1}(X) := \mathcal{G}(X \times (\mathbb{A}^1 - 0))/\mathcal{G}(X)$ is also a homotopy invariant presheaf of abelian groups with $\mathbb{Z}F_*$ -transfers. If the presheaf \mathcal{G} is a Nisnevich sheaf, then the presheaf \mathcal{F}_{-1} is also a Nisnevich sheaf. If the presheaf \mathcal{G} is stable, then the presheaf \mathcal{G}_{-1} is stable too.

If the presheaf \mathcal{G} is a Nisnevich sheaf on Sm/k and Y is a k -smooth variety, then denote by $\mathcal{G}|_Y$ the restriction of \mathcal{G} to the small Nisnevich site of Y .

Lemma 14.2. The category of Nisnevich sheaves with $\mathbb{Z}F_*$ -transfers is a Grothendieck category.

Lemma 14.3. *For any Nisnevich sheaf \mathcal{F} with $\mathbb{Z}F_*$ -transfers, any integer n and any k -smooth variety X , there is a natural isomorphism*

$$H_{\text{Nis}}^n(X, \mathcal{F}) = \text{Ext}^n(\mathbb{Z}F_*(X), \mathcal{F}),$$

where the Ext-groups are taken in the Grothendieck category of Nisnevich sheaves with $\mathbb{Z}F_*$ -transfers.

Corollary 14.4. *For any Nisnevich sheaf \mathcal{F} with $\mathbb{Z}F_*$ -transfers and any integer n , the presheaf $X \mapsto H_{\text{Nis}}^n(X, \mathcal{F})$ has a canonical structure of a $\mathbb{Z}F_*$ -presheaf.*

In fact, this holds for the presheaf $X \mapsto \text{Ext}^n(\mathbb{Z}F_*(X), \mathcal{F})$.

Lemma 14.5. *For any \mathbb{A}^1 -invariant stable $\mathbb{Z}F_*$ -sheaf of abelian groups \mathcal{F} , any k -smooth variety Y and any k -smooth divisor D in Y the canonical morphism*

$$H_D^1(Y, \mathcal{F}) \rightarrow H_{\text{Nis}}^0(Y, \mathcal{H}_D^1(Y, \mathcal{F}))$$

is an isomorphism.

Proof. The local-global spectral sequence yields an exact sequence of the form

$$H_{\text{Nis}}^1(Y, \mathcal{H}_D^0(Y, \mathcal{F})) \rightarrow H_D^1(Y, \mathcal{F}) \rightarrow H_{\text{Nis}}^0(Y, \mathcal{H}_D^1(Y, \mathcal{F})) \rightarrow H_{\text{Nis}}^2(Y, \mathcal{H}_D^0(Y, \mathcal{F})).$$

By Theorem 2.15(item (3')) the sheaf $\mathcal{H}_D^0(Y, \mathcal{F})$ vanishes. \square

Lemma 14.6. *Let X be an essentially k -smooth local henselian scheme and let $D \subset X$ be a k -smooth divisor. Let $i : D \hookrightarrow X$ be the closed embedding. Then for any \mathbb{A}^1 -invariant stable $\mathbb{Z}F_*$ -sheaf of abelian groups \mathcal{F} the Nisnevich sheaves $i_*(\mathcal{F}_{-1}|_D)$ and $\mathcal{H}_D^1(X, \mathcal{F})$ on the small Nisnevich site of X are isomorphic.*

Proof. The group $H_D^1(X, \mathcal{F})$ is isomorphic to $\mathcal{F}(X - D)/\text{Im}(\mathcal{F}(X)) = \mathcal{F}(X - D)/\mathcal{F}(X)$. The latter equality makes sense by Theorem 2.15(item (3')). Since X is essentially k -smooth and henselian, and D is essentially k -smooth, then there is a morphism $r : X \rightarrow D$ such that the composite map $D \xrightarrow{i} X \xrightarrow{r} D$ is the identity. Let $x \in X$ be the closed point. Clearly, $x \in D$. Set $V := \text{Spec}(\mathcal{O}_{D \times \mathbb{A}^1, (x, 0)})$. Let $f \in k[X]$ be a function defining the smooth divisor D . Then the morphism

$$(r, f) : X \rightarrow D \times \mathbb{A}^1$$

takes values in V . We keep the same notation for the corresponding morphism $(r, f) : X \rightarrow V$. Note that $(r, f)^{-1}(D \times 0) = D$. Thus the morphism (r, f) induces a homomorphism

$$[[(r, f)]]^* : \mathcal{F}(V - D \times 0)/\mathcal{F}(V) \rightarrow \mathcal{F}(X - D)/\mathcal{F}(X).$$

We claim that it is an isomorphism. To prove this claim note that the morphism (r, f) induces a scheme isomorphism $X_D^h \rightarrow V_{D \times 0}^h$, where X_D^h is the henselization of X at D and $V_{D \times 0}^h$ is the henselization of V at $D \times 0$. Now Theorem 2.15(item (5)) implies the claim.

By Corollary 2.16 the pull-back map

$$\mathcal{F}_{-1}(D) = \mathcal{F}(D \times (\mathbb{A}^1 - 0))/\mathcal{F}(D \times \mathbb{A}^1) \rightarrow \mathcal{F}(V - D \times 0)/\mathcal{F}(V)$$

is an isomorphism, too. Thus there is a natural isomorphism

$$\begin{aligned} \mathcal{F}_{-1}(D) &= \mathcal{F}(D \times (\mathbb{A}^1 - 0))/\mathcal{F}(D \times \mathbb{A}^1) \xrightarrow{[[\text{can}]]^*} \mathcal{F}(V - D)/\mathcal{F}(V) \xrightarrow{[[(r, f)]]^*} \mathcal{F}(X - D)/\mathcal{F}(X) = \\ &= \mathcal{F}(X - D)/\text{Im}(\mathcal{F}(X)) \xrightarrow{\cong} H_D^1(X, \mathcal{F}) \end{aligned}$$

leading to an isomorphism of Nisnevich sheaves $i_*(\mathcal{F}_{-1}|_D) \cong \mathcal{H}_D^1(X, \mathcal{F})$ on the small Nisnevich site of X . \square

Lemma 14.7. *Let X be an essentially k -smooth local henzelian scheme and let $D \subset X$ be a k -smooth divisor. Let $I : D \times \mathbb{A}^1 \hookrightarrow X \times \mathbb{A}^1$ be the closed embedding. Then for any \mathbb{A}^1 -invariant stable $\mathbb{Z}F_*$ -sheaf of abelian groups \mathcal{F} the two Nisnevich sheaves $I_*(\mathcal{F}_{-1}|_{D \times \mathbb{A}^1})$ and $\mathcal{H}_{D \times \mathbb{A}^1}^1(X \times \mathbb{A}^1, \mathcal{F})$ on small Nisnevich site of $X \times \mathbb{A}^1$ are isomorphic.*

Proof. The proof is similar to that of Lemma 14.6. \square

Corollary 14.8. *The pull-back map $p_X^* : H_D^1(X, \mathcal{F}) \rightarrow H_{D \times \mathbb{A}^1}^1(X \times \mathbb{A}^1, \mathcal{F})$ is an isomorphism, where $p_X : X \times \mathbb{A}^1 \rightarrow X$ is the projection.*

Proof. One can check that the following diagram commutes

$$\begin{array}{ccccc}
 \mathcal{F}_{-1}(D \times \mathbb{A}^1) & \xrightarrow{\beta} & H_{Nis}^0(X \times \mathbb{A}^1, \mathcal{H}_{D \times \mathbb{A}^1}^1(X \times \mathbb{A}^1, \mathcal{F})) & \xleftarrow{\psi} & H_{D \times \mathbb{A}^1}^1(X \times \mathbb{A}^1, \mathcal{F}) \\
 p_D^* \uparrow & & p_X^* \uparrow & & p_X^* \uparrow \\
 \mathcal{F}_{-1}(D) & \xrightarrow{\alpha} & H_{Nis}^0(X, \mathcal{H}_D^1(X, \mathcal{F})) & \xleftarrow{\varphi} & H_D^1(X, \mathcal{F})
 \end{array}$$

where the maps α and β are isomorphisms of Lemmas 14.6 and 14.7 respectively, the maps φ and ψ are isomorphisms of Lemma 14.5, the vertical maps are the pull-back maps. The sheaf \mathcal{F}_{-1} is homotopy invariant, because so is the sheaf \mathcal{F} . It follows that the map p_D^* is an isomorphism, whence the corollary. \square

Corollary 14.9. *Under the the hypotheses of Lemma 14.7 the boundary map*

$$\partial : \mathcal{F}((X - D) \times \mathbb{A}^1) \rightarrow H_{D \times \mathbb{A}^1}^1(X \times \mathbb{A}^1, \mathcal{F})$$

is surjective.

Proof. By Corollary 14.8 the pull-back map $p_X^* : H_D^1(X, \mathcal{F}) \rightarrow H_{D \times \mathbb{A}^1}^1(X \times \mathbb{A}^1, \mathcal{F})$ is an isomorphism. The boundary map $\partial : \mathcal{F}(X - D) \rightarrow H_D^1(X, \mathcal{F})$ is surjective since X is local henselian, whence the corollary. \square

Proposition 14.10. *Under the the hypotheses of Lemma 14.7 the map*

$$H_{Nis}^1(X \times \mathbb{A}^1, \mathcal{F}) \rightarrow H_{Nis}^1((X - D) \times \mathbb{A}^1, \mathcal{F})$$

is injective.

Proof. This follows from Corollary 14.9. \square

Proposition 14.11. *Let \mathcal{F} be an \mathbb{A}^1 -invariant stable $\mathbb{Z}F_*$ -sheaf of abelian groups. Then*

$$H_{Nis}^1(\mathbb{A}_K^1, \mathcal{F}) = 0.$$

Proof. Let $a \in H_{Nis}^1(\mathbb{A}_K^1, \mathcal{F})$. We want to prove that $a = 0$. The Nisnevich topology is trivial at the generic point of the affine line \mathbb{A}_K^1 . Therefore there is a Zariski open subset U in \mathbb{A}_K^1 such that the restriction of a to U vanishes. Let Z be the complement of U in \mathbb{A}_K^1 regarded as a closed subscheme with the reduced structure (it consists of finitely many closed points). Let $V := \sqcup_{z \in Z} (\mathbb{A}^1)_z^h$, where each summand is the henselization of the affine line at $z \in Z$. Then the cartesian square

$$\begin{array}{ccc}
 V - Z & \longrightarrow & V \\
 \downarrow & & \downarrow \Pi \\
 U & \longrightarrow & \mathbb{A}_K^1
 \end{array}$$

gives rise to a long exact sequence

$$\mathcal{F}(U) \oplus \mathcal{F}(V) \rightarrow \mathcal{F}(V - Z) \xrightarrow{\partial} H_{Nis}^1(\mathbb{A}_K^1, \mathcal{F}) \rightarrow H_{Nis}^1(U, \mathcal{F}) \oplus H_{Nis}^1(V, \mathcal{F}).$$

The left arrow is surjective by Theorem 2.15 (items (2), (5)). The group $H_{Nis}^1(V, \mathcal{F})$ vanishes by the choice of V . Thus the map $H_{Nis}^1(\mathbb{A}_K^1, \mathcal{F}) \rightarrow H_{Nis}^1(U, \mathcal{F})$ is injective, and hence $a = 0$. \square

Proposition 14.12. *Suppose the base field k is infinite and perfect. Let \mathcal{F} be an \mathbb{A}^1 -invariant stable $\mathbb{Z}F_*$ -sheaf of abelian groups. Let Y be a k -smooth variety and let $a \in H_{Nis}^1(Y \times \mathbb{A}^1, \mathcal{F})$ be an element such that its restriction to $X \times \{0\}$ vanishes. Then $a = 0$.*

Proof. The exact sequence

$$0 \rightarrow H_{Nis}^1(Y, p_*(\mathcal{F})) \xrightarrow{\alpha} H_{Nis}^1(Y \times \mathbb{A}^1) \xrightarrow{\beta} H_{Nis}^0(Y, R^1 p_*(\mathcal{F}))$$

and the fact that the sheaf \mathcal{F} is homotopy invariant show that for

$$A := \text{Ker}[i_0^* : H_{Nis}^1(Y \times \mathbb{A}^1, \mathcal{F}) \rightarrow H_{Nis}^1(Y, \mathcal{F})]$$

the map $\beta|_A : A \rightarrow H_{Nis}^0(Y, R^1 p_*(\mathcal{F}))$ is injective. The stalk of the sheaf $R^1 p_*(\mathcal{F})$ at a point $y \in Y$ is $H_{Nis}^1(Y_y^h \times \mathbb{A}^1, \mathcal{F})$, where $Y_y^h = \text{Spec}(\mathcal{O}_{Y,y}^h)$ is the henselization of the local scheme $\text{Spec}(\mathcal{O}_{Y,y})$. By Proposition 14.11 there is a closed subset Z in Y such that $\beta(a)|_{Y-Z} = 0$. Since the field k is perfect, there is a proper closed subset $Z_1 \subset Z$ such that $Z - Z_1$ is k -smooth. Then $Z - Z_1$ is a k -smooth closed subvariety in $Y - Z_1$.

We claim that $a_1 := a|_{(Y-Z_1) \times \mathbb{A}^1} = 0$. In fact, $a_1|_{(Y-Z_1) \times 0} = 0$. Thus it suffices to check that all stalks of the element $\beta(a_1)$ vanish. Let $y \in Y - Z_1$ be a point. If $y \in Y - Z$ then $\beta(a_1)_y = 0$, because $\beta(a)|_{Y-Z} = 0$. If $y \in Z - Z_1$ then shrinking $Y - Z_1$ around y we may assume that there is a k -smooth divisor D in $Y - Z_1$ containing $Z - Z_1$. In this case $a_1|_{Y-D} = 0$. Now Proposition 14.10 shows that $\beta(a_1)_y = 0$. We have proved that $a_1 = 0$.

Now there is a proper closed subset $Z_2 \subset Z_1$ such that $Z_1 - Z_2$ is k -smooth. Then $Z_1 - Z_2$ is a k -smooth closed subvariety in $Y - Z_2$. Arguing just as above, we conclude that $a_2 := a|_{(Y-Z_2) \times \mathbb{A}^1} = 0$. Continuing this process finitely many times, we conclude that $a = 0$. \square

Theorem 14.13. *Suppose the base field k is infinite and perfect. If \mathcal{F} is an \mathbb{A}^1 -invariant stable $\mathbb{Z}F_*$ -sheaf of abelian groups, then the $\mathbb{Z}F_*$ -presheaf of abelian groups $X \mapsto H_{Nis}^1(X, \mathcal{F})$ is \mathbb{A}^1 -invariant and stable.*

Proof. By Corollary 14.4 the presheaf $X \mapsto H_{Nis}^1(X, \mathcal{F})$ has a canonical structure of a $\mathbb{Z}F_*$ -presheaf. Let X be a k -smooth variety. Let $\sigma_X \in \text{Fr}_1(X, X)$ be the distinguished morphism of level one. The assignment $X \mapsto (\sigma_X^* : \mathcal{F}(X) \rightarrow \mathcal{F}(X))$ is an endomorphism of the Nisnevich sheaf $\mathcal{F}|_{\text{Sm}/k}$. Thus for each n it induces an endomorphism of the cohomology presheaf $\sigma^* : H^n(-, \mathcal{F}) \rightarrow H^n(-, \mathcal{F})$. Since σ^* acts on \mathcal{F} as the identity, it acts as the identity on the presheaf $H^n(-, \mathcal{F})$. We see that the $\mathbb{Z}F_*$ -presheaf $H^n(-, \mathcal{F})$ is stable.

To show that the presheaf $X \mapsto H_{Nis}^1(X, \mathcal{F})$ is \mathbb{A}^1 -invariant, note that the pull-back map $i_0^* : H_{Nis}^1(X \times \mathbb{A}^1, \mathcal{F}) \rightarrow H_{Nis}^1(X, \mathcal{F})$ is surjective. It is also injective by Proposition 14.12. Our theorem now follows. \square

Theorem 14.14. *Suppose the base field k is infinite and perfect. Let \mathcal{F} be an \mathbb{A}^1 -invariant stable $\mathbb{Z}F_*$ -sheaf of abelian groups. Then for any integer $n \geq 2$, the presheaf $X \mapsto H_{Nis}^n(X, \mathcal{F})$ is an \mathbb{A}^1 -invariant and stable $\mathbb{Z}F_*$ -presheaf of abelian groups.*

Proof. We can apply the same arguments as in the proof of Theorem 14.13 to show that the presheaf $X \mapsto H_{Nis}^n(X, \mathcal{F})$ is a $\mathbb{Z}F_*$ -presheaf of abelian groups, which is, moreover, stable.

It remains to check that the presheaf is homotopy invariant. We may assume till the end of the proof that each presheaf $X \mapsto H_{Nis}^j(X, \mathcal{F})$ with $j < n$ is homotopy invariant.

In order to complete the proof of the theorem, we shall need a couple of lemmas. The first lemma is as follows.

Lemma 14.15. *For any \mathbb{A}^1 -invariant stable $\mathbb{Z}F_*$ -sheaf of abelian groups \mathcal{F} , any k -smooth variety Y , any k -smooth divisor D in Y and any integer $n \geq 2$, one has the equality $H_D^n(Y, \mathcal{F}) = 0$.*

Proof. Applying the local-global spectral sequence, it is sufficient to check that the groups $H_{Nis}^i(X \times \mathbb{A}^1, \mathcal{H}_{D \times \mathbb{A}^1}^j(X \times \mathbb{A}^1, \mathcal{F}))$ vanish for $i + j = n$ (here $i, j \geq 0$). Let $j = 1$. By Lemma 14.7 one has $\mathcal{H}_{D \times \mathbb{A}^1}^1(X \times \mathbb{A}^1, \mathcal{F}) = I_*(\mathcal{F}_{-1})$. Thus,

$$\begin{aligned} H_{Nis}^{n-1}(X \times \mathbb{A}^1, \mathcal{H}_{D \times \mathbb{A}^1}^1(X \times \mathbb{A}^1, \mathcal{F})) &= H_{Nis}^{n-1}(X \times \mathbb{A}^1, I_*(\mathcal{F}_{-1})) = \\ &= H_{Nis}^{n-1}(D \times \mathbb{A}^1, \mathcal{F}_{-1}) = H_{Nis}^{n-1}(D, \mathcal{F}_{-1}) = 0. \end{aligned}$$

The latter equality holds since D is local henzelian. If $j = 0$, then the sheaf $\mathcal{H}_{D \times \mathbb{A}^1}^j(X \times \mathbb{A}^1, \mathcal{F})$ vanishes. Obviously, the stalk of this sheaf vanishes at every point $z \in (X - D) \times \mathbb{A}^1$. If $z \in D \times \mathbb{A}^1$ then the stalk vanishes by Theorem 2.15 (item (3')). Let $2 \leq j \leq n$. Then the sheaf $\mathcal{H}_{D \times \mathbb{A}^1}^j(X \times \mathbb{A}^1, \mathcal{F})$ vanishes. Obviously, the stalk of this sheaf vanishes at every point $z \in (X - D) \times \mathbb{A}^1$. If $z \in D \times \mathbb{A}^1$ then the stalk of the sheaf at the point z is equal to the group $H_{\mathcal{D}}^j(\mathcal{X}, \mathcal{F})$, where \mathcal{X} is the henselization of $X \times \mathbb{A}^1$ at z and \mathcal{D} is the henselization of $D \times \mathbb{A}^1$ at z . Since $H_{Nis}^j(\mathcal{X}, \mathcal{F}) = 0$ one has equalities

$$H_{\mathcal{D}}^j(\mathcal{X}, \mathcal{F}) = H_{Nis}^{j-1}(\mathcal{X} - \mathcal{D}, \mathcal{F}) / \text{Im}[H_{Nis}^{j-1}(\mathcal{X}, \mathcal{F})] = H_{Nis}^{j-1}(\mathcal{X} - \mathcal{D}, \mathcal{F}) / H_{Nis}^{j-1}(\mathcal{X}, \mathcal{F}).$$

The latter equality makes sense by Theorem 2.15(item (3')).

Now, applying Theorem 2.15(item (5)) and Corollary 2.16 to the presheaf $H_{Nis}^{j-1}(-, \mathcal{F})$ and arguing as in the proof of Lemma 14.6 we get an isomorphism

$$H_{Nis}^{j-1}(\mathcal{X} - \mathcal{D}, \mathcal{F}) / H_{Nis}^{j-1}(\mathcal{X}, \mathcal{F}) \cong H_{Nis}^{j-1}(\mathcal{D} \times \mathbb{G}_m, \mathcal{F}) / H_{Nis}^{j-1}(\mathcal{D} \times \mathbb{A}^1, \mathcal{F}).$$

It suffices to check that $H_{Nis}^{j-1}(\mathcal{D} \times \mathbb{G}_m, \mathcal{F}) = 0$. The presheaf $H_{Nis}^{j-1}(- \times \mathbb{G}_m, \mathcal{F})$ is a homotopy invariant $\mathbb{Z}F_*$ -presheaf, which is also stable. By Theorem 2.15(item (3')) the map

$$H_{Nis}^{j-1}(\mathcal{D} \times \mathbb{G}_m, \mathcal{F}) \rightarrow H_{Nis}^{j-1}(\text{Spec}(k(D)) \times \mathbb{G}_m, \mathcal{F}) = H_{Nis}^{j-1}(\mathbb{G}_{m, k(D)}, \mathcal{F})$$

is injective. By Theorem 2.9 the map $H_{Nis}^{j-1}(\mathbb{G}_{m, k(D)}, \mathcal{F}) \rightarrow H_{Nis}^{j-1}(\text{Spec}(k(D)(t)), \mathcal{F})$ is injective. The latter group vanishes, because the Nisnevich topology on $\text{Spec}(K)$ is trivial, where K is a finitely generated field over k . Thus $H_{Nis}^{j-1}(\mathcal{D} \times \mathbb{G}_m, \mathcal{F}) = 0$ and $H_{\mathcal{D}}^j(\mathcal{X}, \mathcal{F}) = 0$, too. The lemma follows. \square

Let us return to the proof of Theorem 14.14. Under the hypotheses of Lemma 14.7, the preceding lemma implies the map

$$H_{Nis}^n(X \times \mathbb{A}^1, \mathcal{F}) \rightarrow H_{Nis}^n((X - D) \times \mathbb{A}^1, \mathcal{F}) \quad (32)$$

is injective.

Next, we claim that for a k -smooth variety Y and the projection $p : Y \times \mathbb{A}^1 \rightarrow Y$ the Nisnevich sheaves $R^j p_*(\mathcal{F})$ vanish for $j = 1, \dots, n - 1$. In fact, such a sheaf is associated with

the presheaf $U \mapsto H_{Nis}^j(U \times \mathbb{A}^1, \mathcal{F})$. The presheaf $H_{Nis}^j(U, \mathcal{F})$ is homotopy invariant. Thus $H_{Nis}^j(U \times \mathbb{A}^1, \mathcal{F}) = H_{Nis}^j(U, \mathcal{F})$. Since $j \geq 1$ the associated Nisnevich sheaf vanishes.

Since the Nisnevich sheaves $R^j p_*(\mathcal{F})$ vanish for $j = 1, \dots, n-1$, one has an exact sequence

$$0 \rightarrow H_{Nis}^n(Y, p_*(\mathcal{F})) \xrightarrow{\alpha} H_{Nis}^n(Y \times \mathbb{A}^1) \xrightarrow{\beta} H_{Nis}^0(Y, R^n p_*(\mathcal{F})).$$

Since the sheaf \mathcal{F} is homotopy invariant, then for

$$A := \text{Ker}[i_0^* : H_{Nis}^n(Y \times \mathbb{A}^1, \mathcal{F}) \rightarrow H_{Nis}^1(Y, \mathcal{F})]$$

the map $\beta|_A : A \rightarrow H_{Nis}^0(Y, R^n p_*(\mathcal{F}))$ is injective. Arguing as in the proof of Proposition 14.12 and using the fact that map (32) is injective, we get the following

Lemma 14.16. *Suppose the base field k is infinite and perfect. Let \mathcal{F} be an \mathbb{A}^1 -invariant stable $\mathbb{Z}F_*$ -sheaf of abelian groups. Let Y be a k -smooth variety and let $a \in H_{Nis}^n(Y \times \mathbb{A}^1, \mathcal{F})$ be an element such that its restriction to $Y \times \{0\}$ vanishes. Then $a = 0$.*

The pull-back map $i_0^* : H_{Nis}^n(Y \times \mathbb{A}^1, \mathcal{F}) \rightarrow H_{Nis}^n(Y, \mathcal{F})$ is surjective by functoriality. By Lemma 14.16 it is also injective. This completes the proof of Theorem 14.14. \square

REFERENCES

- [1] M. Artin, Comparaison avec la cohomologie classique: cas d'un préschéma lisse, in Théorie des topos et cohomologie étale des schémas (SGA 4). Tome 3. Lect. Notes Math., vol. 305, Exp. XI, Springer-Verlag, Berlin-New York, 1973.
- [2] A. Druzhinin, On the homotopy invariant presheaves with Witt-transfers, Russian Math Surveys, 2014, 69:3, 575–577.
- [3] G. Garkusha, I. Panin, Framed motives of algebraic varieties (after V. Voevodsky), preprint arXiv:1409.4372.
- [4] I. Panin, A. Stavrova, N. Vavilov, Grothendieck–Serre’s conjecture concerning principal G -bundles over reductive group schemes: I, Compos. Math. 151(3) (2015), 535–567.
- [5] A. Suslin, V. Voevodsky, Singular homology of abstract algebraic varieties, Invent. Math. 123(1) (1996), 61–94.
- [6] V. Voevodsky, Notes on framed correspondences, unpublished, 2001 (also available at www.math.ias.edu/vladimir/files/framed.pdf).
- [7] V. Voevodsky, Cohomological theory of presheaves with transfers, in Cycles, Transfers, and Motivic Homology Theories, Ann. Math. Studies, 2000, Princeton University Press.
- [8] M. Walker, Motivic cohomology and the K-theory of automorphisms, PhD Thesis, University of Illinois at Urbana-Champaign, 1996.

DEPARTMENT OF MATHEMATICS, SWANSEA UNIVERSITY, SINGLETON PARK, SWANSEA SA2 8PP, UNITED KINGDOM

E-mail address: g.garkusha@swansea.ac.uk

ST. PETERSBURG BRANCH OF V. A. STEKLOV MATHEMATICAL INSTITUTE, FONTANKA 27, 191023 ST. PETERSBURG, RUSSIA

E-mail address: paniniv@gmail.com

INSTITUTE FOR ADVANCED STUDY, EINSTEIN DRIVE, PRINCETON, NJ, 08540, USA

E-mail address: panin@math.ias.edu